# DYNAMIC BEHAVIOURS OF BLACK HOLE PHASE TRANSITIONS NEAR QUADRUPLE POINTS

4/29/2026


Speaker: Jiayue Yang
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PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

Based on "Jiayue Yang and Robert B. Mann. Dynamic behaviours of black hole phase transitions near quadruple points. JHEP, 08:028, 2023."

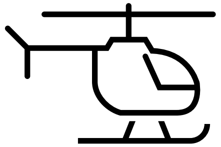
# Outline

- Introduction
- Quadruple points in black hole phase transitions
- Dynamic processes at the quadruple points
- The first passage events
- Conclusions

# INTRODUCTION

# Introduction

- What are black holes?



How fast would you need to travel to escape the gravitational pull of the earth?

# Introduction

- What are black holes?

How fast would you need to travel to escape the gravitational pull of the earth?

11.2km per second!

# Introduction

- What are black holes？

- Denser planets/stars → larger escape velocity

➢ Dark Star Proposal

In 1783, John Michell realized that the gravity of a star could be strong enough that the escape velocity might exceed the speed of light.

In this case, light cannot emit by this star and we could not see the star. He named it "dark star".

# Introduction

- What are black holes?

➢ Schwarzschild Solution

In 1916, Karl Schwarzschild obtained an exact solution to Einstein's field equation, the spherically symmetric Schwarzschild solution.

This is a space-time region with a radius called the event horizon (Schwarzschild radius) that has such strong gravity that nothing can escape, not even light.

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

$$f(r) = 1 - \frac{2M}{r}$$

# Introduction


Credit: EHT Collaboration

- Black holes actually exist

➢ In 2002, Genzel et al. found the first indication that black hole may exist. They found that there exists a supermassive black hole at the center of the Milky Way.

Ref: https://arxiv.org/abs/astro-ph/0210426

➢ On September 14, 2015 (GW150914), the Laser Interferometer Gravitational-Wave Observatory ( LIGO) collaboration first directly detected gravitational waves from the merger of two black holes.

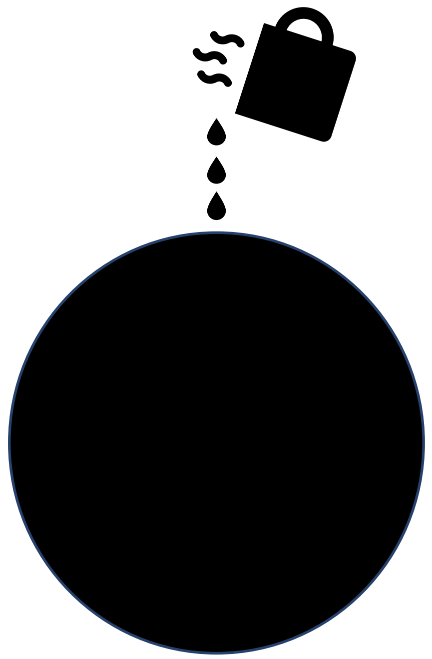
Ref: https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.116.061102

➢ In 2019, the Event Horizon Telescope (EHT) collaboration took the first-ever image of a black hole, which lies at the center of the M87 galaxy. This is the first time we humans have "seen" a black hole.

Ref: https://iopscience.iop.org/article/10.3847/2041-8213/ab0ec7

# Introduction

**John Wheeler: What would happen if you pour a cup of hot coffee into the black hole?**



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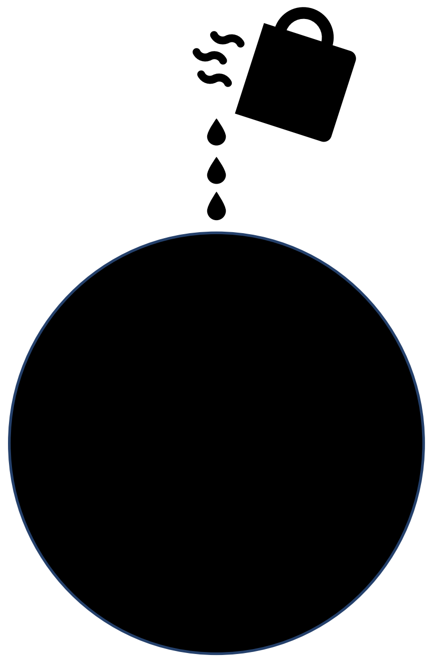
**John Wheeler: What would happen if you pour a cup of hot coffee into the black hole?**

Hot coffee has entropy!
Black holes swallow hot coffee.
But where does the entropy go?



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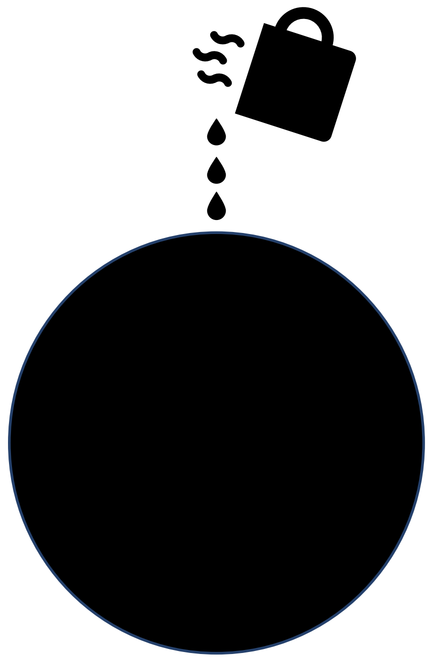

**Jacob Bekenstein: Black holes must have entropy!**

When black holes swallow coffee, the mass increases, the horizon expands outward, the area of the horizon increases!

black hole entropy ~ horizon area

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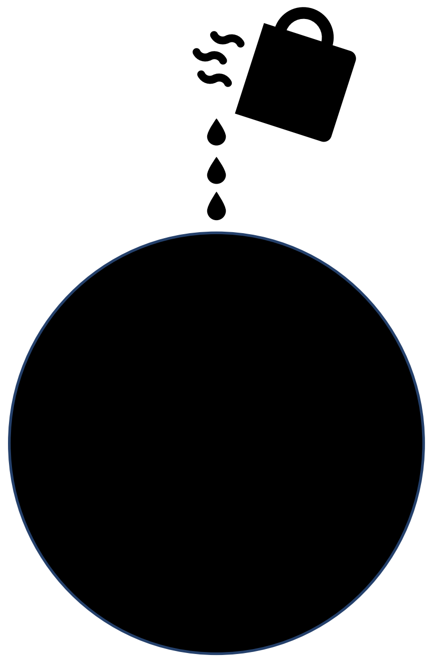

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The horizon area~ black hole entropy

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Letter | Published: 01 March 1974

**Black hole explosions?**

S. W. HAWKING

*Nature* **248**, 30–31 (1974) | Cite this article

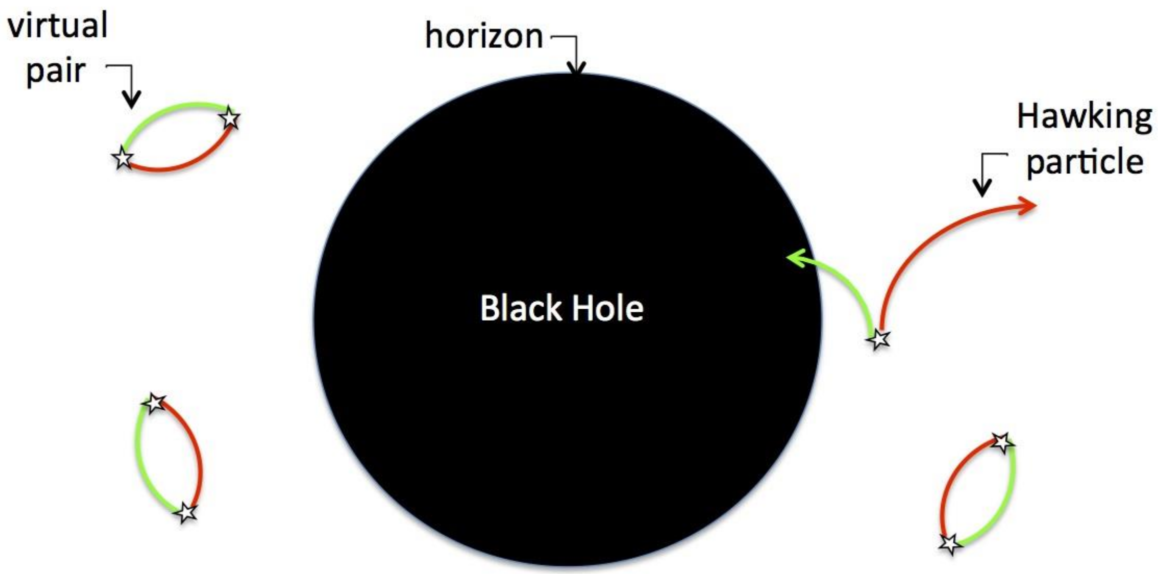


Commun. math. Phys. 43, 199—220 (1975)


Particle Creation by Black Holes

S. W. Hawking
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge, England

Received April 12, 1975




https://blogsofmalik.blogspot.com/2018/03/an-inspirational-story-stephen-hawking.html

Hawking radiation's spectrum is the same as that of thermal radiation. The corresponding temperature is what we call *Hawking temperature*.



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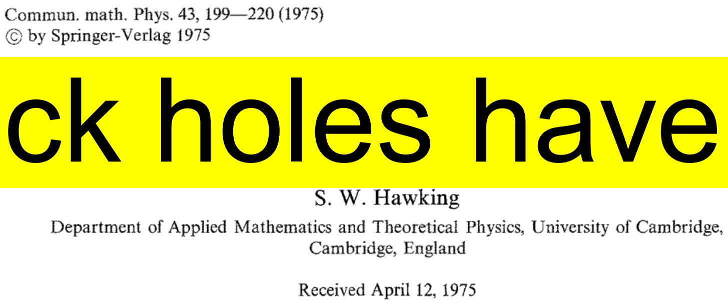
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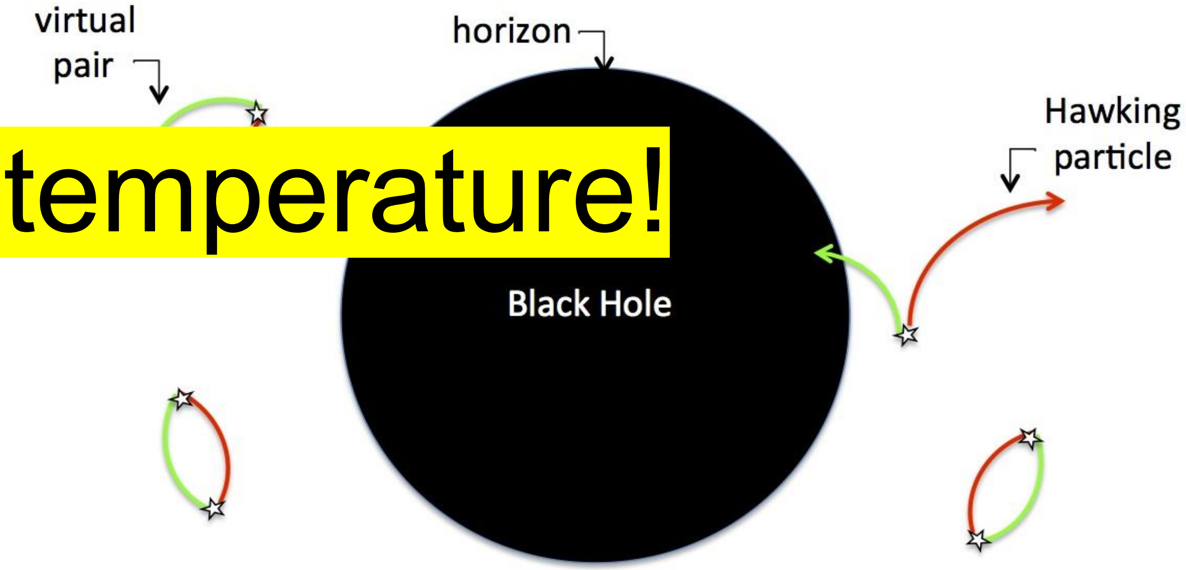
**Black holes have temperature!**

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# Introduction

- Four Laws of Black Hole Mechanics (Bardeen, Carter, and Hawking)

Commun. math. Phys. 31, 161–170 (1973)


## The Four Laws of Black Hole Mechanics

J. M. Bardeen*
Department of Physics, Yale University, New Haven, Connecticut, USA

B. Carter and S. W. Hawking
Institute of Astronomy, University of Cambridge, England

Received January 24, 1973

# Introduction

- Four Laws of Black Hole Mechanics (Bardeen, Carter, and Hawking)

0. The surface gravity $\kappa$ is constant over the event horizon of a stationary black hole.

Kubizňák, D., Mann, R. B., & Teo, M. (2017). Black hole chemistry: thermodynamics with Lambda. *Classical and Quantum Gravity*, *34*(6), 063001.

[arXiv:1608.06147](arXiv:1608.06147)

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1. For a rotating charged black hole with a mass M, an angular momentum J, and a charge Q

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega\delta J + \Phi\delta Q$$

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# Introduction

- Four Laws of Black Hole Mechanics (Bardeen, Carter, and Hawking)

0. The surface gravity κ is constant over the event horizon of a stationary black hole. κ ~T

1. For a rotating charged black hole with a mass M, an angular momentum J, and M ~E

a charge Q

$$\delta M = \frac{\kappa}{8\pi G}\delta A + \Omega \delta J + \Phi \delta Q$$

2. Hawking's area theorem: δA ≥ 0, i.e. the area A of a black hole's event horizon can never decrease. A ~S

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- Four Laws of Black Hole Mechanics (Bardeen, Carter, and Hawking)

0. The surface gravity κ is constant over the event horizon of a stationary black hole. $\kappa \sim T$

1. For a rotating charged black hole with a mass M, an angular momentum J, and a charge Q **Black holes have thermodynamics!** $M \sim E$

2. Hawking's area theorem: $\delta A \geq 0$, i.e. the area A of a black hole's event horizon can never decrease. $A \sim S$

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# Introduction

- From Black Hole Thermodynamics
- To Black Hole Chemistry?

The next few slides are based on: Kubizňák, D., Mann, R. B., & Teo, M. (2017). Black hole chemistry: thermodynamics with Lambda. *Classical and Quantum Gravity*, *34*(6), 063001.

[arXiv:1608.06147](arXiv:1608.06147)

# Introduction-Black Hole Chemistry

- From Black Hole Thermodynamics to Black Hole Chemistry?

$$\delta E = T\delta S$$
$$\delta M = T\delta S$$

$$\delta H = T\delta S + V\delta P$$
$$\delta M = T\delta S + V\delta P$$

Black hole chemistry: thermodynamics with cosmological constant Lambda.

| Thermodynamics | | Black Hole Mechanics | |
|---|---|---|---|
| Enthalpy | $H = E + PV$ | Mass | $M$ |
| Temperature | $T$ | Surface Gravity | $\frac{\kappa}{2\pi}$ |
| Entropy | $S$ | Horizon Area | $\frac{A}{4}$ |
| Pressure | $P$ | Cosmological Constant | $-\frac{\Lambda}{8\pi}$ |
| First Law | $\delta H = T\delta S + V\delta P + \ldots$ | First Law | $\delta M = \frac{\kappa}{8\pi}\delta A + V\delta P + \ldots$ |

Kubizňák, D., Mann, R. B., & Teo, M. (2017). Black hole chemistry: thermodynamics with Lambda. *Classical and Quantum Gravity*, *34*(6), 063001.

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# Introduction-Black Hole Chemistry

- Main idea of black hole chemistry: identify the mass as enthalpy (instead of energy) cosmological constant and its conjugate variable as pressure and volume, respectively.

- Mass→Enthalpy
- Enthalpy=the energy to create the BH +the energy to place the BH in the environment that has pressure P
- $H = E + PV$

- Cosmological constant→ Pressure
- Positive $\Lambda$, cosmic tension, (fluid that has negative pressure) dS!
- Negative $\Lambda$, cosmic pressure (fluid that has positive pressure)! AdS!

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This is black hole chemistry!

- Cosmological constant→ Pressure
- Positive Λ, cosmic tension, (fluid that has negative pressure) dS!
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arXiv:1608.06147

# Introduction

- Black Hole Phase Transition----Hawking–Page transition

Schwarzschild-AdS Solution

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2}$$

Thermodynamic quantities

$$M = \frac{r_+}{2}(1 + \frac{r_+^2}{l^2}), \qquad S = \pi r_+^2, \qquad T = \frac{l^2 + 3r_+^2}{4\pi l^2 r_+}, \qquad P = \frac{3}{8\pi l^2}, \qquad V = \frac{4\pi r_+^3}{3}$$

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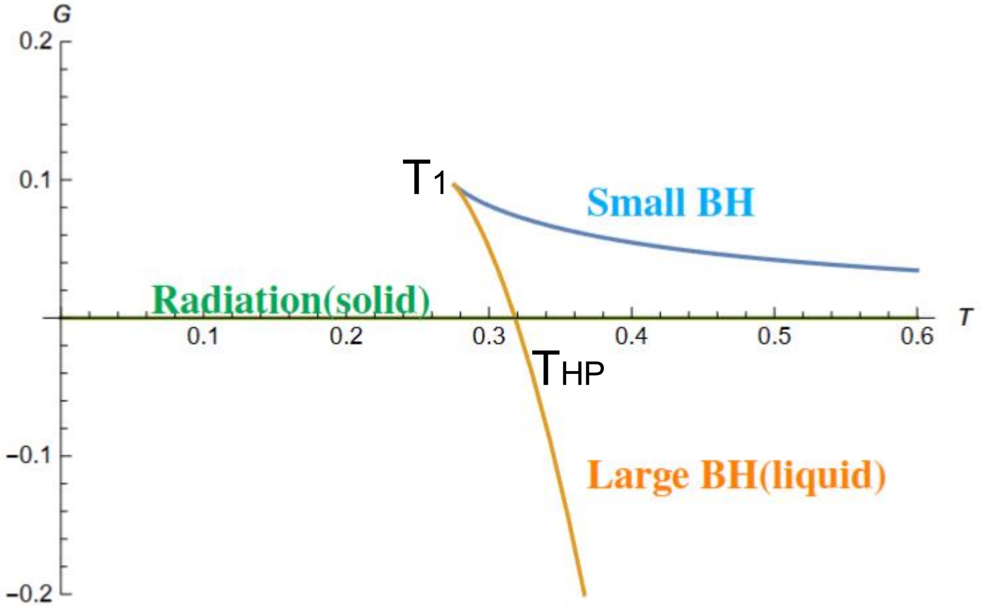
# Introduction

- Black Hole Phase Transition----Hawking–Page transition

Gibbs free energy

$$
\begin{aligned}
G &= M - TS \\
&= \frac{r_+}{2} + \frac{r_+^3}{2l^2} - \frac{r_+}{4}(1 + 3\frac{r_+^2}{l^2}) \\
&= \frac{2l^2 r_+^2 + 2r_+^4 - l^2 r_+^2 - 3r_+^4}{4l^2 r_+} \\
&= \frac{l^2 r_+^2 - r_+^4}{4l^2 r_+} \\
&= \frac{l^2 r_+ - r_+^3}{4l^2}
\end{aligned}
$$

$$T = \frac{l^2 + 3r_+^2}{4\pi l^2 r_+}$$




First order phase transition between the thermal radiation and large black holes at $T_{HP}$

Kubizňák, D., Mann, R. B., & Teo, M. (2017). Black hole chemistry: thermodynamics with Lambda. *Classical and Quantum Gravity*, *34*(6), 063001.

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# Introduction

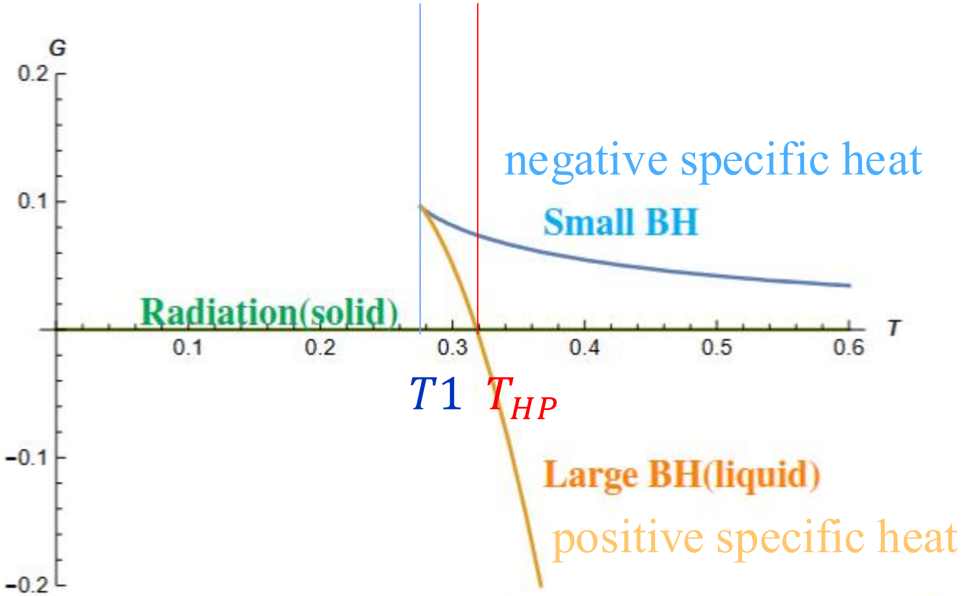


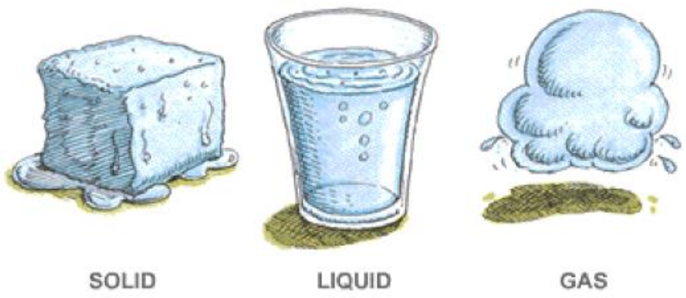

**Figure 3.1:** Hawking–Page transition (Blue curve is small BH, orange curve is large BH, Green curve is radiation) at fixed $P = \frac{3}{8\pi}$

1. $T \leq T_1$(the intersection of Small BH and Large BH), no black hole can exist. It is just the thermal AdS spacetime.
2. $T_1 \leq T \leq T_{HP}$ Black holes are unstable and will emit Hawking radiation to gradually change to thermal AdS spacetime.
3. $T \geq T_{HP}$ Preferred phase are large black holes.There is a first-order phase transition between thermal radiation and large black holes.

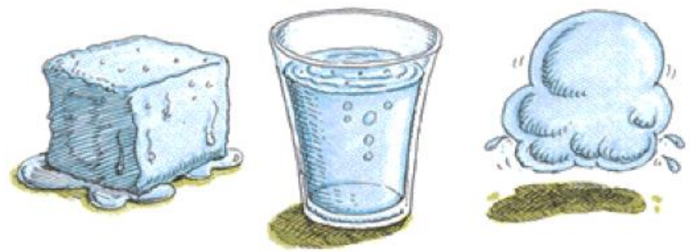
arXiv:1608.06147

# Background-Black Hole Triple Points

➢ Water has triple points, where three phases can coexist.



# Background-Black Hole Triple Points




- Water has triple points, where three phases can coexist.

- What about the black holes?

- Can black holes have triple points, where three black hole phases can coexist?

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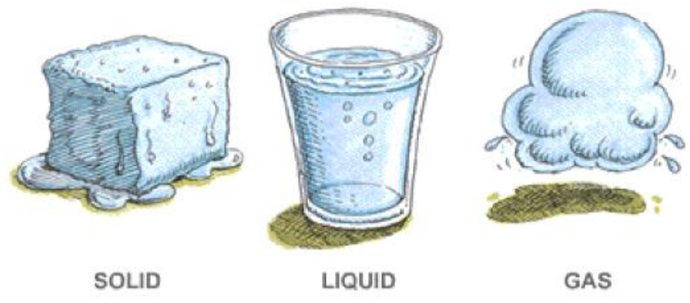



- What about the black holes?


- Can black holes have triple points, where three black hole phases can coexist?

- Phases of black holes are distinguished and characterized by the sizes of BHs.

# Background-Black Hole Triple Points

Black Hole Triple Points were found!

- Charged Gauss-Bonnet black holes in AdS space

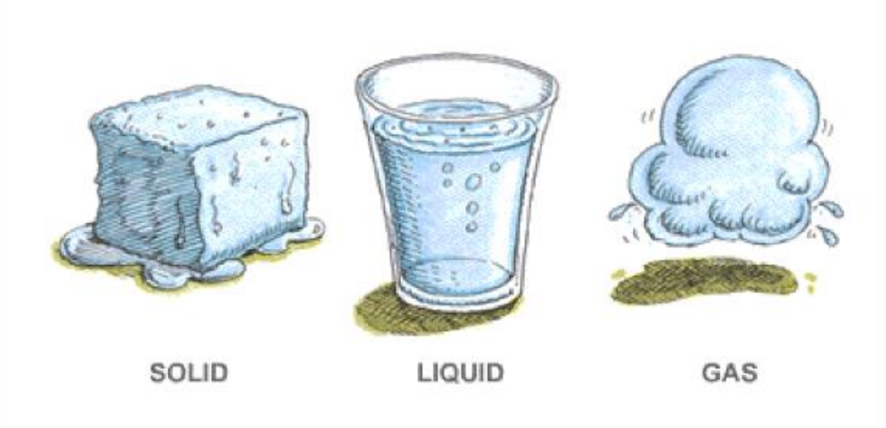
Ref: Wei, S. W., & Liu, Y. X. (2014). Triple points and phase diagrams in the extended phase space of charged Gauss-Bonnet black holes in AdS space. *Physical Review D*, *90*(4), 044057.

- Multi-spinning d=6 Kerr-anti de Sitter black holes

Ref: Altamirano, N., Kubizňák, D., Mann, R. B., & Sherkatghanad, Z. (2014). Kerr-AdS analogue of triple point and solid/liquid/gas phase transition. *Classical and Quantum Gravity*, *31*(4), 042001.

- Lovelock black holes

Ref: Frassino, A. M., Kubizňák, D., Mann, R. B., & Simovic, F. (2014). Multiple reentrant phase transitions and triple points in Lovelock thermodynamics. *Journal of High Energy Physics*, *2014*(9), 1-47.





Phases of black holes are distinguished and characterized by the sizes of BHs.

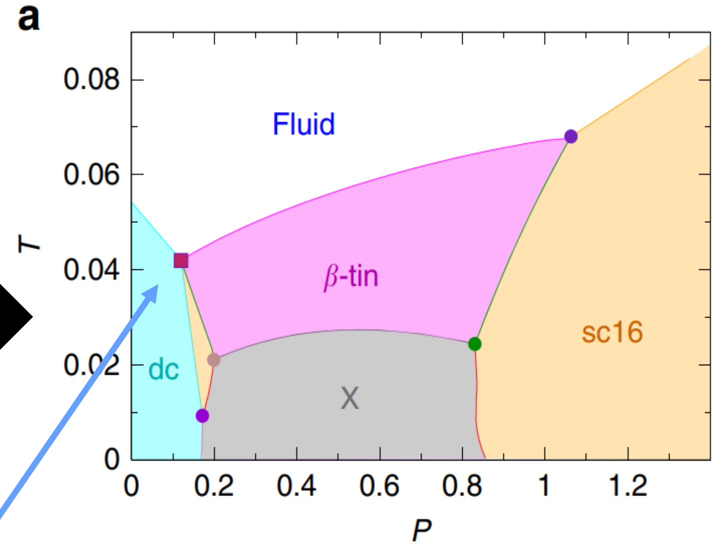
arXiv:1402.2837 arXiv:1308.2672 arXiv:1406.7015

# Background-Black Hole Quadruple Point

Quadruple points: points where 4 phases coexist!

- ➢ Quadruple points were observed in polymers and colloids.

Ref: Akahane, K., Russo, J., & Tanaka, H. (2016). A possible four-phase coexistence in a single-component system. Nature communications, 7(1), 12599.




quadruple point

[arXiv:1611.07148](arXiv:1611.07148) [arXiv:2207.03505](arXiv:2207.03505)

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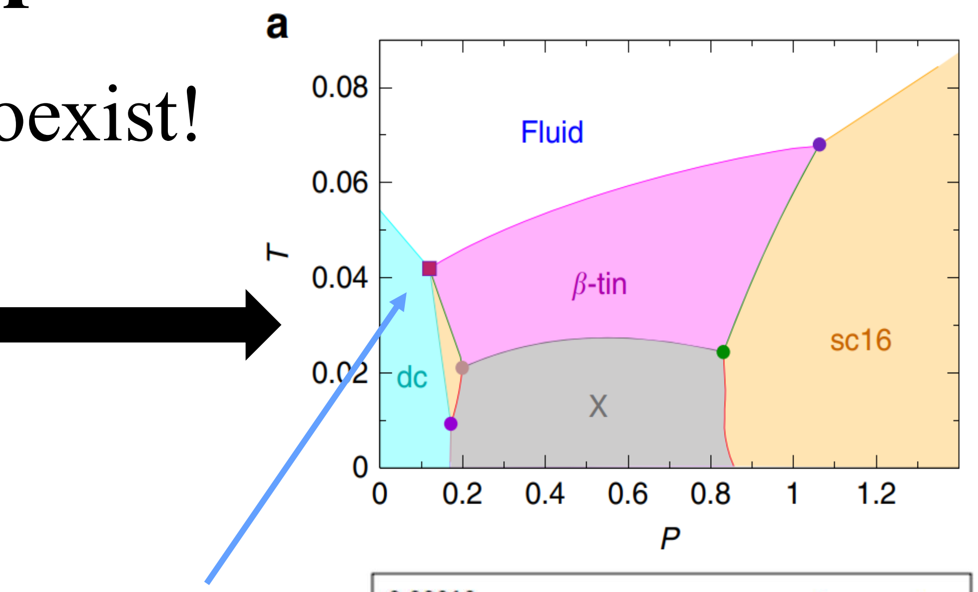
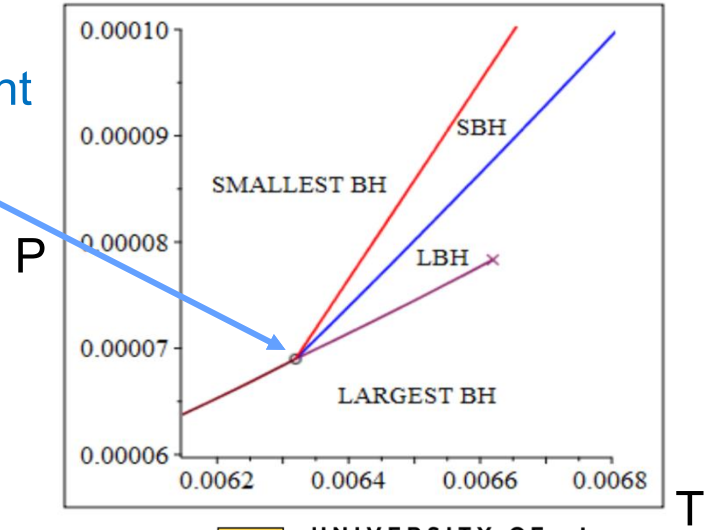



- Question: can we also have black hole quadruple points? (four black hole phases coexist!)

Ref: Tavakoli, M., Wu, J. & Mann, R.B. Multi-critical points in black hole phase transitions. J. High Energ. Phys. 2022, 117 (2022).




[arXiv:1611.07148](arXiv:1611.07148) [arXiv:2207.03505](arXiv:2207.03505)

# QUADRUPLE POINTS IN BLACK HOLE PHASE TRANSITIONS

# Set-up

The action of four-dimensional Einstein gravity coupled with nonlinear electrodynamics (NLE) read

$$S=\int d^4x[\sqrt{-g}(R-2\Lambda-\sum_{i=1}^{N}\alpha_i(F^2)^i)]$$

$$F^2 = F_{\mu\nu}F^{\mu\nu}$$

where $F\mu\nu$ is electromagnetic field tensor, $\alpha_i$ are coupling constants

By varying the action, we get Einstein Power-Maxwell equations

$$G_{\mu\nu}=-2\frac{dL_{EM}}{dF^2}F_\mu{}^\lambda F_{\nu\lambda}+\frac{1}{2}g_{\mu\nu}L_{EM}$$

$$\nabla_\mu\left(\frac{dL_{EM}}{dF^2}F^{\mu\nu}\right)=0$$

$$L_{EM}=-\sum_{i=1}^{N}\alpha_i(F^2)^i$$

arXiv:2106.13486

# Set-up

Consider the following ansatz describing spherically symmetric static black holes

$$ds^2 = -U(r)dt^2 + \frac{1}{U(r)}dr^2 + r^2 d\Omega^2$$

$$A_\mu = [\Phi(r), 0, 0, 0]$$

yields the below field equations

$$(r(U(r)-1))' + r^2\Lambda$$
$$-r^2 \sum_{n=1}^{N} \left(n - \frac{1}{2}\right) \alpha_n \left(-2(\Phi')^2\right)^n = 0$$
$$\frac{1}{2} r^2 \sum_{n=1}^{N} n\alpha_n \left(-2(\Phi')^2\right)^n - Q(\Phi') = 0$$

arXiv:2106.13486

Series $\Phi = \sum_{r=1}^{K} b_i r^{-i} \qquad U = 1 + \sum_{i=1}^{K} c_i r^{-i} + \frac{r^2}{l^2}$

Solution:

$$c_1 = -2M \qquad c_i = \frac{4Q}{i+2} b_{i-1}, \qquad \text{for} \qquad i > 1$$

$$b_1 = Q \qquad b_5 = \frac{4}{5} Q^3 \alpha_2 \qquad b_9 = \frac{4}{3} Q^5 (4\alpha_2^2 - \alpha_3)$$

$$b_{13} = \frac{32}{13} Q^7 (24\alpha_2^3 - 12\alpha_3\alpha_2 + \alpha_4)$$

$$b_{17} = \frac{80}{17} Q^9 (176\alpha_2^4 - 132\alpha_2^2\alpha_3 + 16\alpha_4\alpha_2 + 9\alpha_3^2 - \alpha_5)$$

$$b_{21} = \frac{64}{7} Q^{11} (1456\alpha_2^5 + 234\alpha_3^2\alpha_2 + 208\alpha_4\alpha_2^2 - 24\alpha_4\alpha_3 - 1456\alpha_2^3\alpha_3 - 20\alpha_5\alpha_2 + \alpha_6)$$

$$\ldots = \ldots\ldots,$$

# Set-up

Focus on a setting where the alphas are nonzero and independent for $i \leq 7$

So we have the below solution: potential $\Phi$ and metric function $U$, with $b_i = 0$ for $i > 25$

$$\Phi = \frac{Q}{r} + \frac{b_5}{r^5} + \frac{b_9}{r^9} + \frac{b_{13}}{r^{13}} + \frac{b_{17}}{r^{17}} + \frac{b_{21}}{r^{21}} + \frac{b_{25}}{r^{25}}$$

$$U = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{b_5 Q}{2r^6} + \frac{b_9 Q}{3r^{10}} + \frac{b_{13} Q}{4r^{14}} + \frac{b_{17} Q}{5r^{18}} + \frac{b_{21} Q}{6r^{22}} + \frac{b_{25} Q}{7r^{26}} + \frac{r^2}{l^2}$$

If all $\alpha_i$ (for $i > 1$) are set to zero, we recover charged AdS black hole

$$\Phi = \frac{Q}{r} \qquad U = \frac{r^2}{l^2} + 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

arXiv:2207.03505

# Set-up

Entropy

$$S = \frac{A}{4} = \pi r_+^2$$

Pressure

$$P = \frac{(d-1)(d-2)}{16\pi l^2} = \frac{3}{8\pi l^2}$$

Volume

$$V = \frac{4\pi r_+^3}{3}$$

Temperature

$$T = \frac{1}{4\pi r_+}\left(1 + \frac{3r_+^2}{l^2} - \frac{Q^2}{r_+^2} - 5\frac{b_5 Q}{2r_+^6} - 3\frac{b_9 Q}{r_+^{10}} - 13\frac{b_{13} Q}{4r_+^{14}} - 17\frac{b_{17} Q}{5r_+^{18}} - 7\frac{b_{21} Q}{2r_+^{22}} - 25\frac{b_{25} Q}{7r_+^{26}}\right)$$

Equation of state

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4} + \frac{5b_5 Q}{16\pi r_+^8} + \frac{3b_9 Q}{8\pi r_+^{12}} + \frac{13b_{13} Q}{32\pi r_+^{16}} + \frac{17b_{17} Q}{40\pi r_+^{20}} + \frac{7b_{21} Q}{16\pi r_+^{24}} + \frac{25b_{25} Q}{56\pi r_+^{28}}$$

arXiv:2207.03505

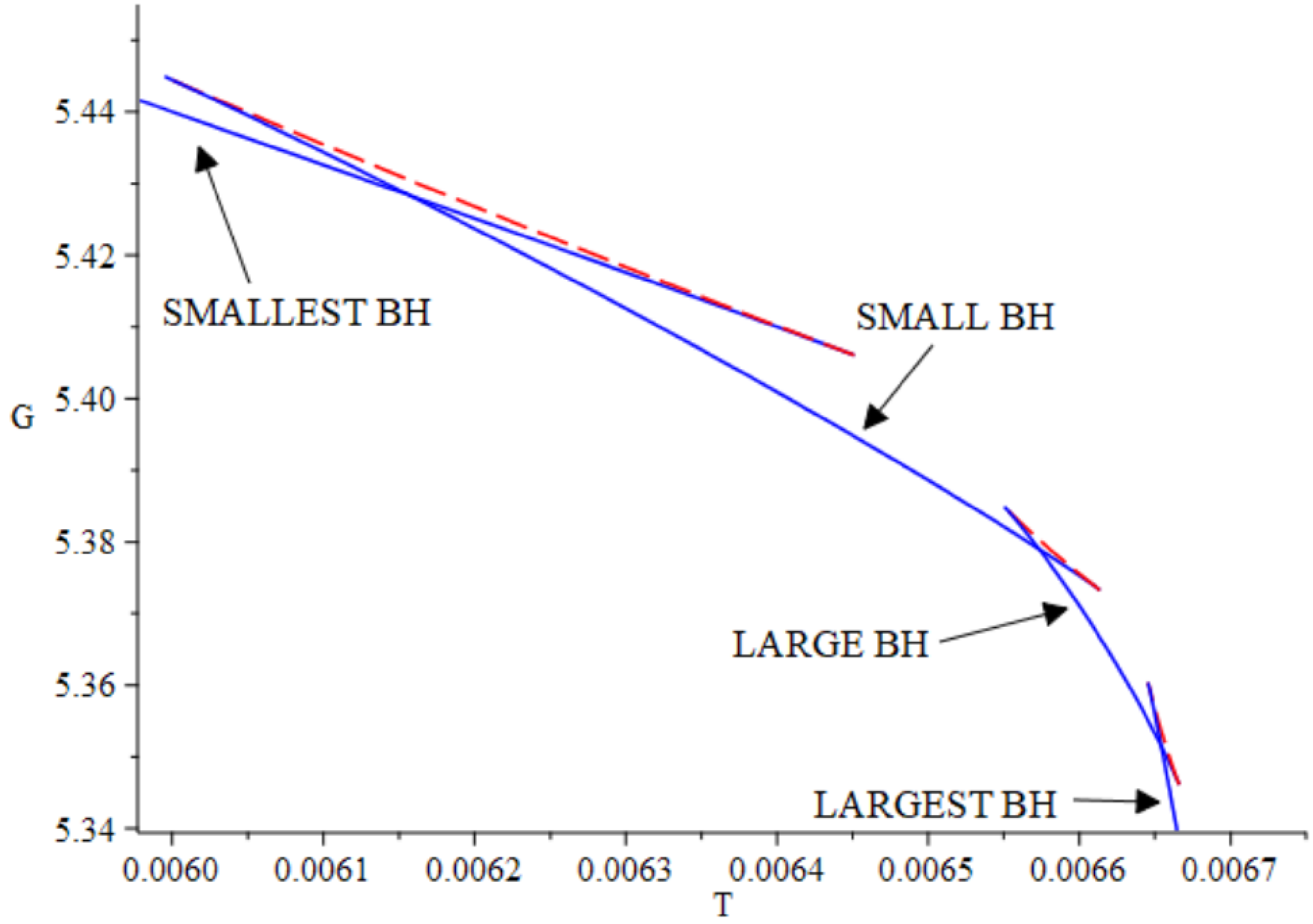
# Quadruple points in black hole phase transitions

Gibbs free energy

$$G = M - TS$$

We observe

- Four black hole phases
- Three swallowtails
- Black hole phase transition



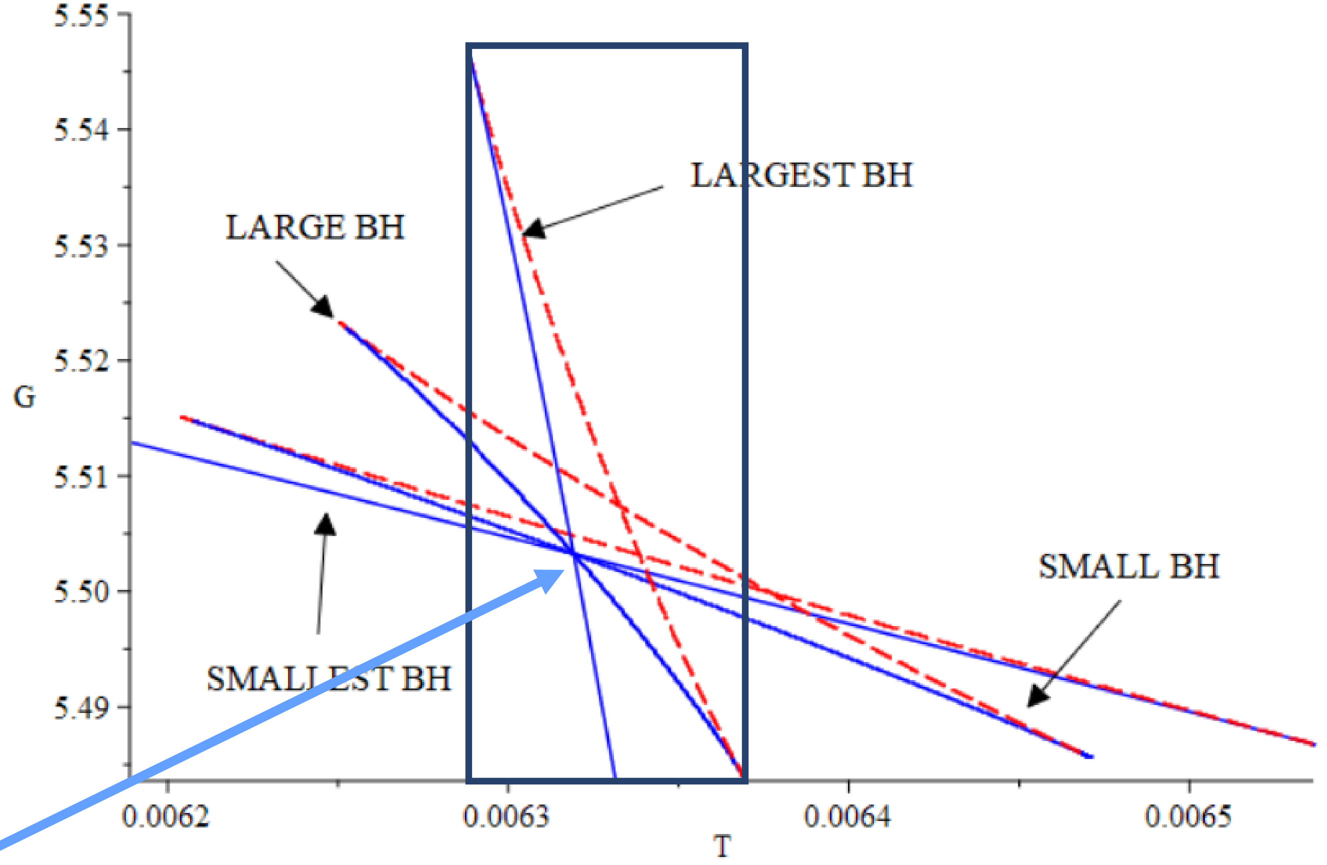

arXiv:2207.03505

# Quadruple points in black hole phase transitions

By adjusting the parameters,

$$\begin{cases} P = 0.0000689999421 \\ Q = 6.751117513 \\ b_5 = -5078.980603 \\ b_9 = 6.054804813 \times 10^6 \\ b_{13} = -4.131510152 \times 10^9 \\ b_{17} = 1.399821234 \times 10^{12} \\ b_{21} = -2.014970449 \times 10^{14} \\ b_{25} = 1.016449472 \times 10^{16} \end{cases}$$

$T \in (0.00628883, 0.00636918)$




In $T \in (0.00628883, 0.00636918)$, 4 phases are permitted!

We found a quadruple point!

arXiv:2207.03505

# Quadruple points in black hole phase transitions

## Off-shell Gibbs free energy

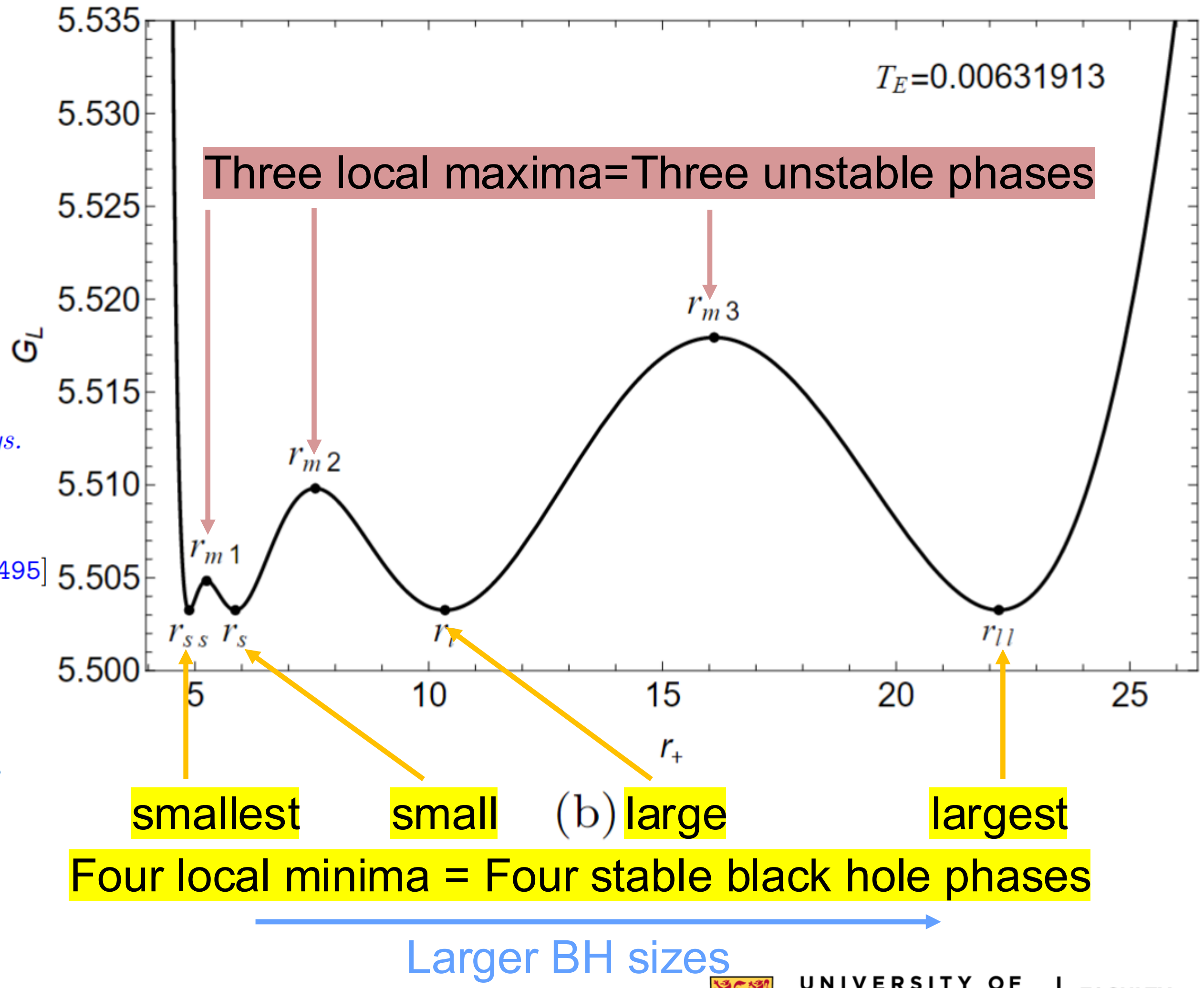
A continuous function of horizon radius at various ensemble temperatures


R. Li and J. Wang, *Thermodynamics and kinetics of Hawking-Page phase transition*, *Phys. Rev. D* **102** (2020) 024085 [INSPIRE].

R. Li, K. Zhang and J. Wang, *Thermal dynamic phase transition of Reissner-Nordström Anti-de Sitter black holes on free energy landscape*, *JHEP* **10** (2020) 090 [arXiv:2008.00495] [INSPIRE].

R. Li and J. Wang, *Energy and entropy compensation, phase transition and kinetics of four dimensional charged Gauss-Bonnet Anti-de Sitter black holes on the underlying free energy landscape*, *Nucl. Phys. B* **976** (2022) 115714 [arXiv:2012.05424] [INSPIRE].

R. Li, K. Zhang and J. Wang, *Probing black hole microstructure with the kinetic turnover of phase transition*, *Phys. Rev. D* **104** (2021) 084076 [arXiv:2102.09439] [INSPIRE].

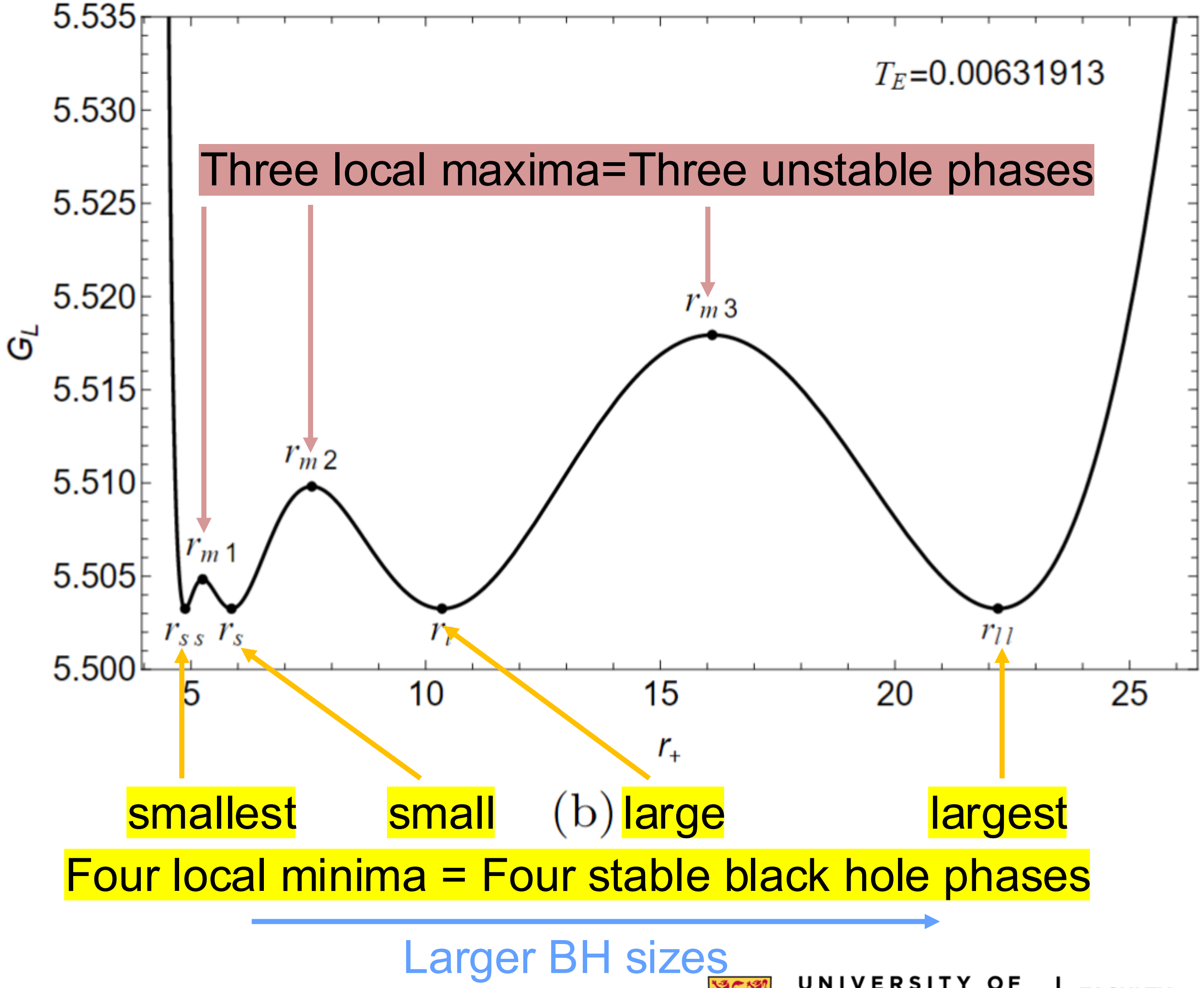



# Quadruple points in black hole phase transitions

## Off-shell Gibbs free energy

A continuous function of horizon radius at various ensemble temperatures

$$
\begin{aligned}
G_{\mathrm{L}} &= M - T_{\mathrm{E}} S \\
&= \frac{1}{840 r_+^{25}} (60 b_{25} Q + 70 b_{21} Q r_+^4 + 84 b_{17} Q r_+^8 \\
&+ 105 b_{13} Q r^{12} + 140 b_9 Q r_+^{16} + 210 b_5 Q r_+^{20} \\
&+ 420 Q^2 r_+^{24} + 420 r_+^{26} + 1120 P \pi r_+^{28}) - T_{\mathrm{E}} \pi r_+^2
\end{aligned}
$$



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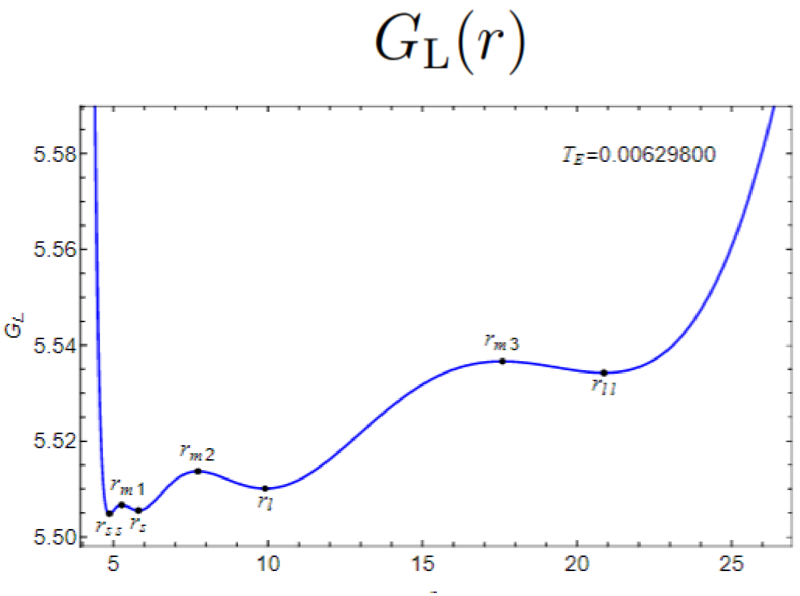
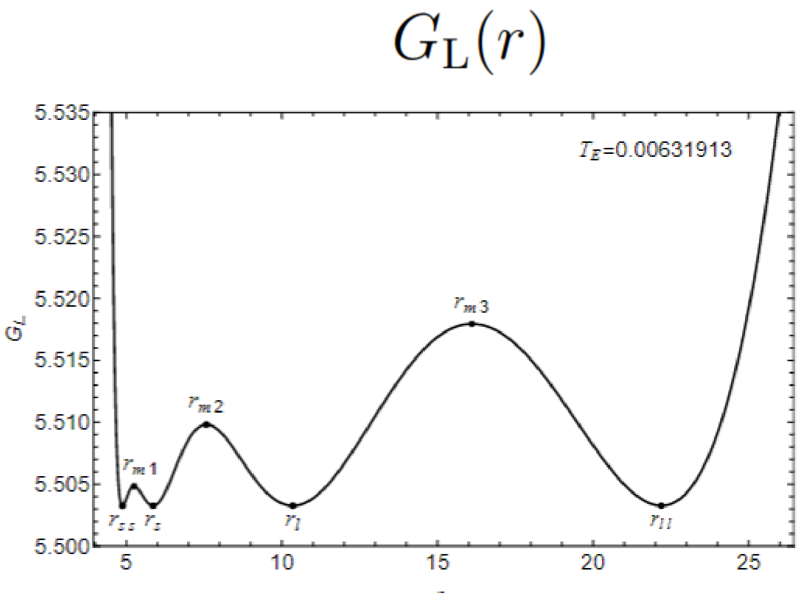
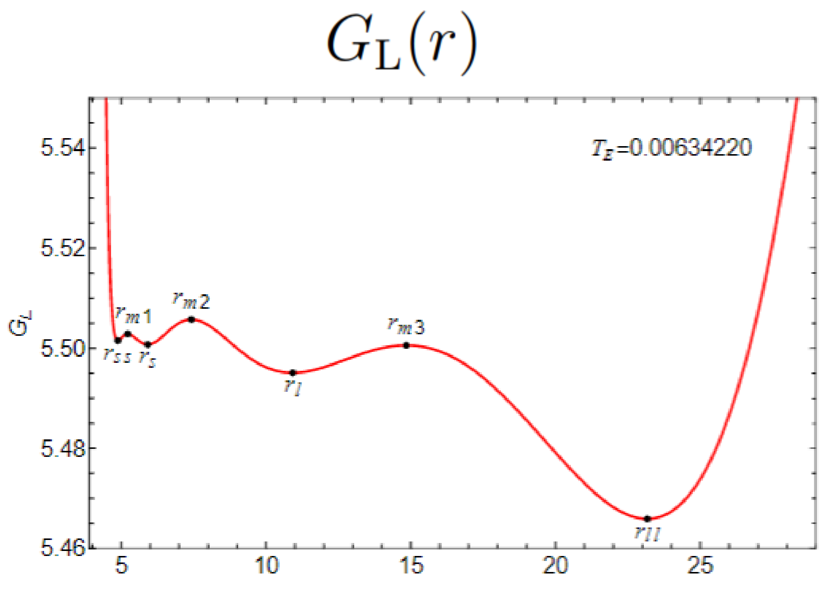
# Off-shell Gibbs free energy vs horizon radius




Larger BH sizes




Larger BH sizes




Larger BH sizes

Cold Temperature
smallest BH is favoured

Quadruple point temperature
No preferred state

Hot temperature
Largest BH is favoured

# DYNAMIC PROCESSES AT THE QUADRUPLE POINT

# Quadruple points

How to understand the dynamics of black hole phase transitions?

Here are some previous studies for your reference.


S.-W. Wei, Y.-X. Liu and Y.-Q. Wang, *Dynamic properties of thermodynamic phase transition for five-dimensional neutral Gauss-Bonnet AdS black hole on free energy landscape*, *Nucl. Phys. B* **976** (2022) 115692 [arXiv:2009.05215] [INSPIRE].

S.-J. Yang, R. Zhou, S.-W. Wei and Y.-X. Liu, *Kinetics of a phase transition for a Kerr-AdS black hole on the free-energy landscape*, *Phys. Rev. D* **105** (2022) 084030 [arXiv:2105.00491] [INSPIRE].

H. Mouri and Y. Taniguchi, *Runaway merging of black holes: analytical constraint on the timescale*, *Astrophys. J. Lett.* **566** (2002) L17 [astro-ph/0201102] [INSPIRE].

A.L. Erickcek, M. Kamionkowski and A.J. Benson, *Supermassive Black Hole Merger Rates: Uncertainties from Halo Merger Theory*, *Mon. Not. Roy. Astron. Soc.* **371** (2006) 1992 [astro-ph/0604281] [INSPIRE].

S.-W. Wei, Y.-Q. Wang, Y.-X. Liu and R.B. Mann, *Observing dynamic oscillatory behavior of triple points among black hole thermodynamic phase transitions*, *Sci. China Phys. Mech. Astron.* **64** (2021) 270411 [arXiv:2102.00799] [INSPIRE].

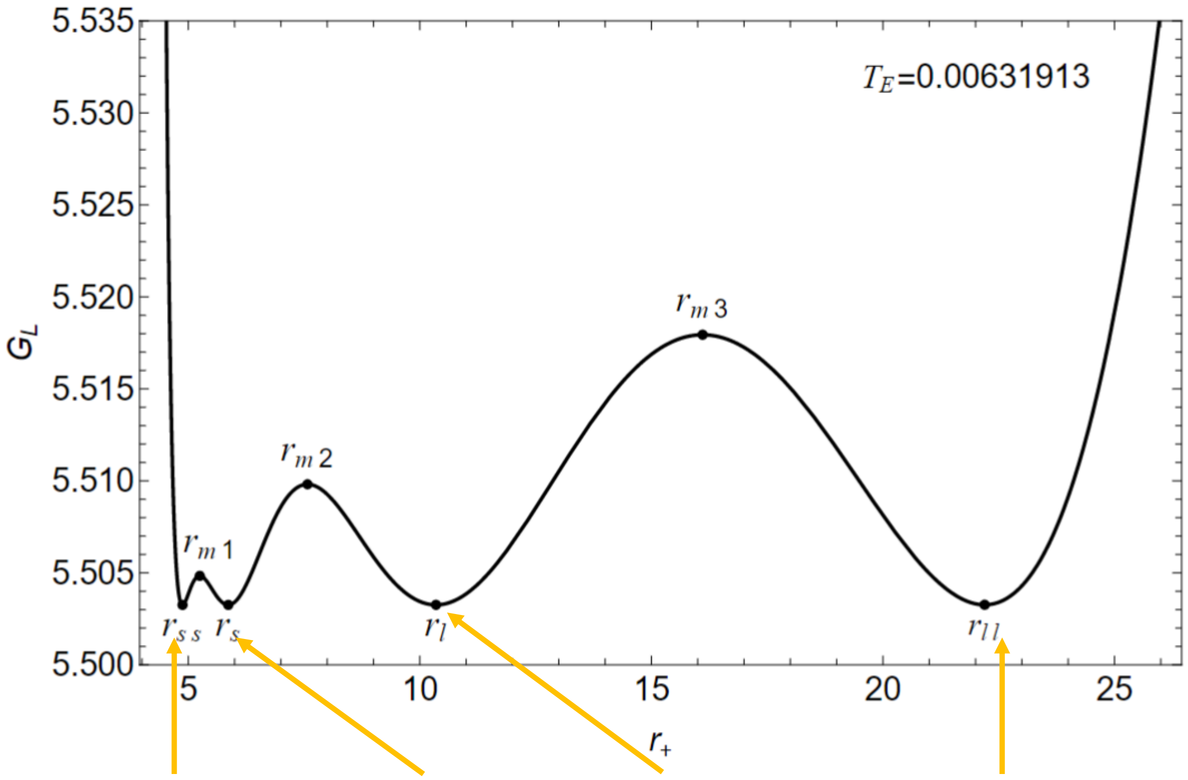
# Quadruple points

How to understand the dynamics of black hole phase transitions?

We need Smoluchowski equation!

$$\frac{\partial \rho(t,r)}{\partial t} = D\frac{\partial}{\partial r}\left(e^{-\beta G_{\mathrm{L}}(r)}\frac{\partial}{\partial r}\left(e^{\beta G_{\mathrm{L}}(r)}\rho(t,r)\right)\right)$$

- $\rho(t,r)$ is the probability density function
- $G_L(r)$ is the off-shell Gibbs free energy
- $\beta$ is the inverse ensemble temperature
- $D$ is the diffusion coefficient



(b)

# Quadruple points

Solve Smoluchowski equation

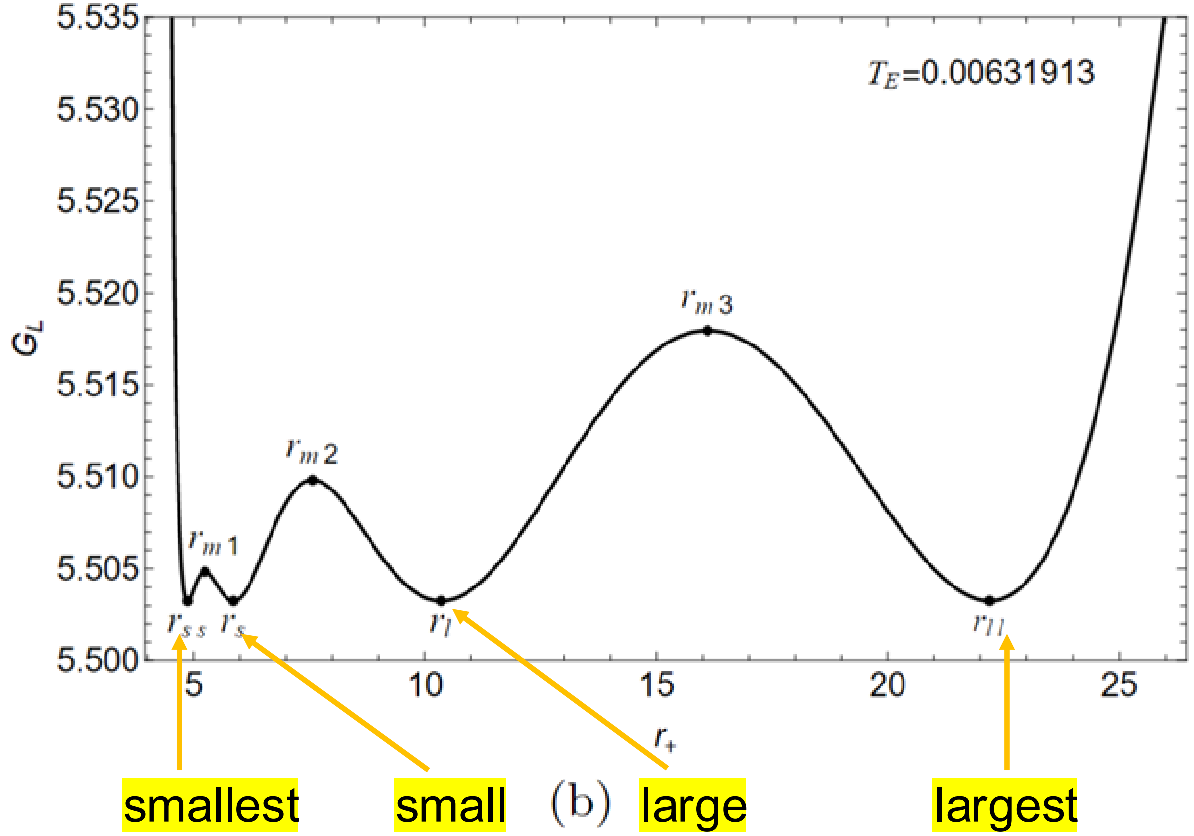
$$\frac{\partial \rho(t,r)}{\partial t} = D\frac{\partial}{\partial r}\left(e^{-\beta G_{\mathrm{L}}(r)}\frac{\partial}{\partial r}\left(e^{\beta G_{\mathrm{L}}(r)}\rho(t,r)\right)\right)$$

- Initial condition

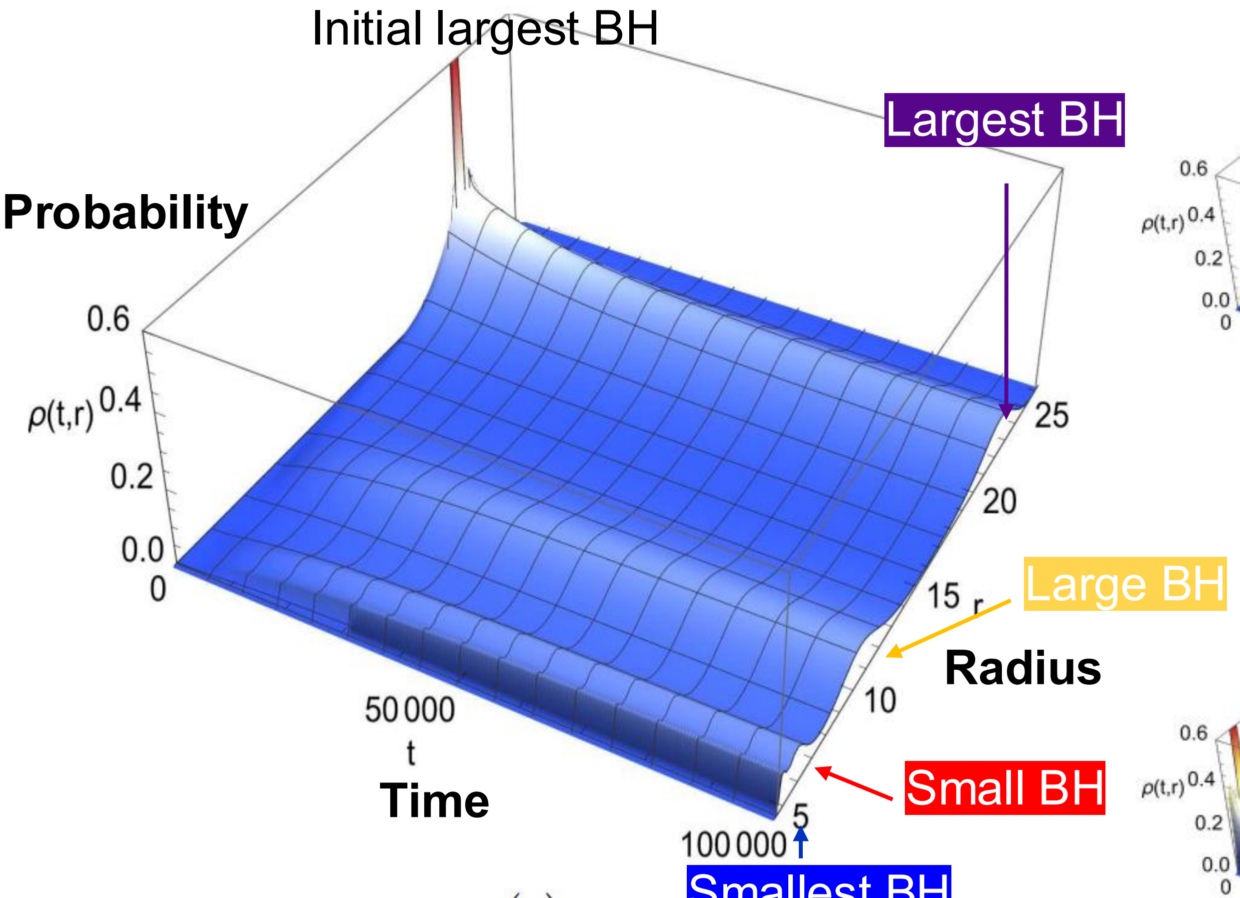
$$\rho(0,r) = \frac{1}{\sigma\sqrt{\pi}}e^{-\frac{(r-r_i)^2}{\sigma^2}} \qquad r_i = r_{ss},\ r_s,\ r_l,\ \text{and } r_{ll}$$

- Boundary condition

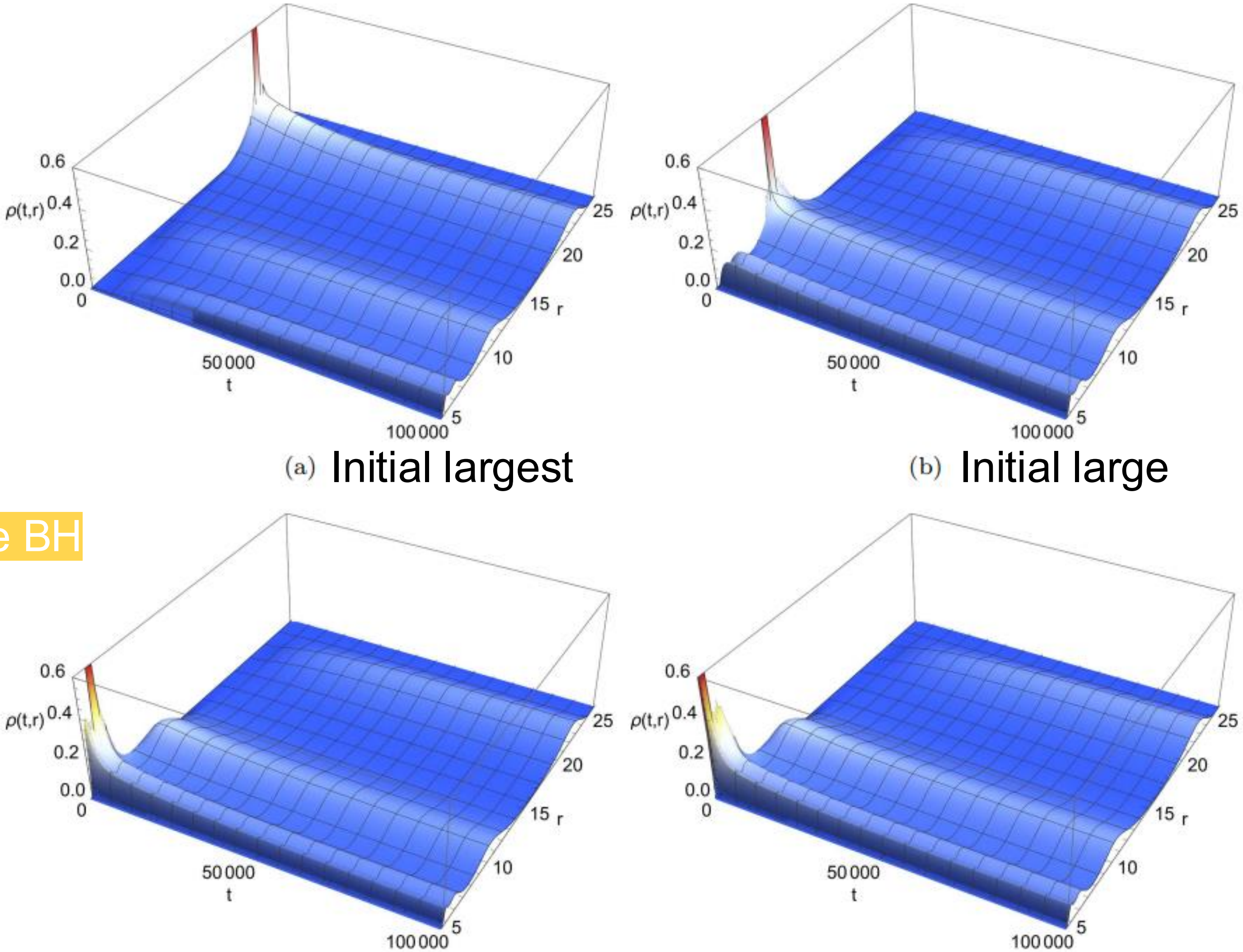
$$e^{-\beta G_{\mathrm{L}}(r)}\frac{\partial}{\partial r}\left(e^{\beta G_{\mathrm{L}}(r)}\rho(t,r)\right)\bigg|_{r=r_{bdy}} = 0$$



# Probability distribution evolution at middle temperature




(a)




(a) Initial largest

(b) Initial large

(c) Initial small

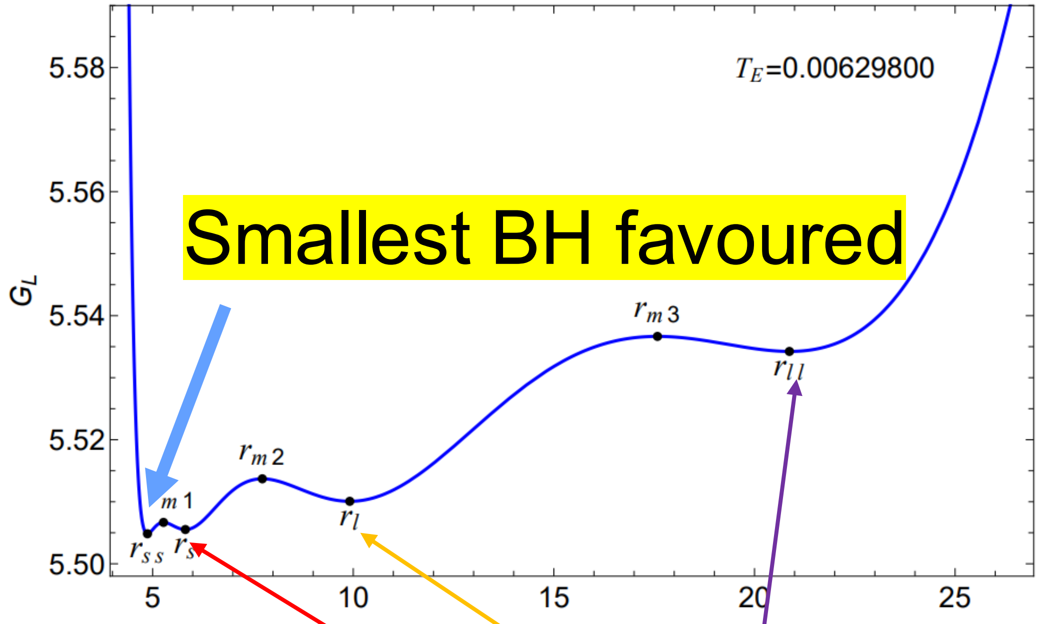
(d) Initial smallest

- Probabilities diffuse from the initial phase to the other three phases.
- Black hole phase transitions happen!

# Probability evolution for 4 phases at low temperature

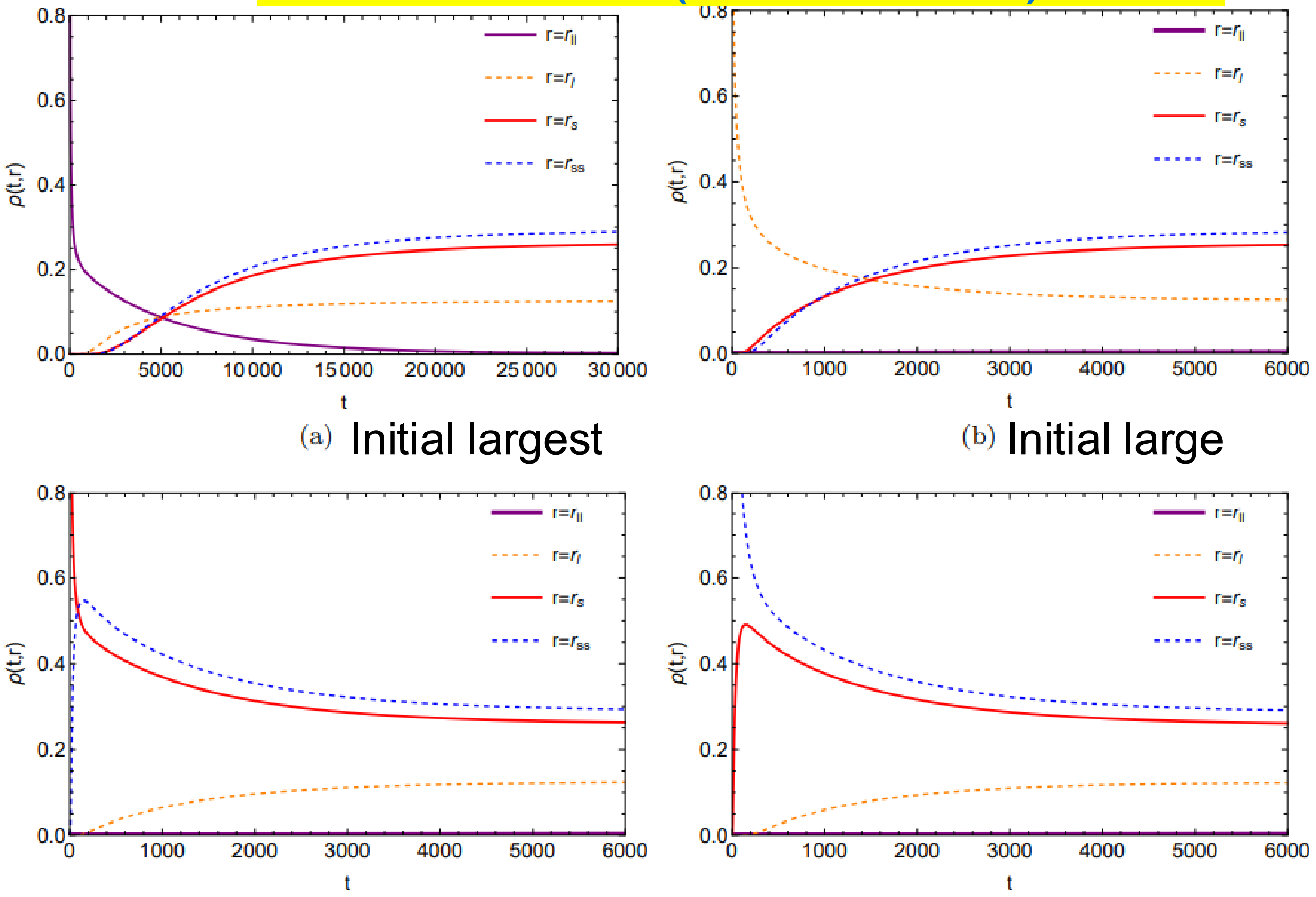
Conclusion:

At low temperature, the smallest BH eventually dominates regardless of the given initial state.




Smallest Small Large Largest

Smallest BH (Blue curve) wins!



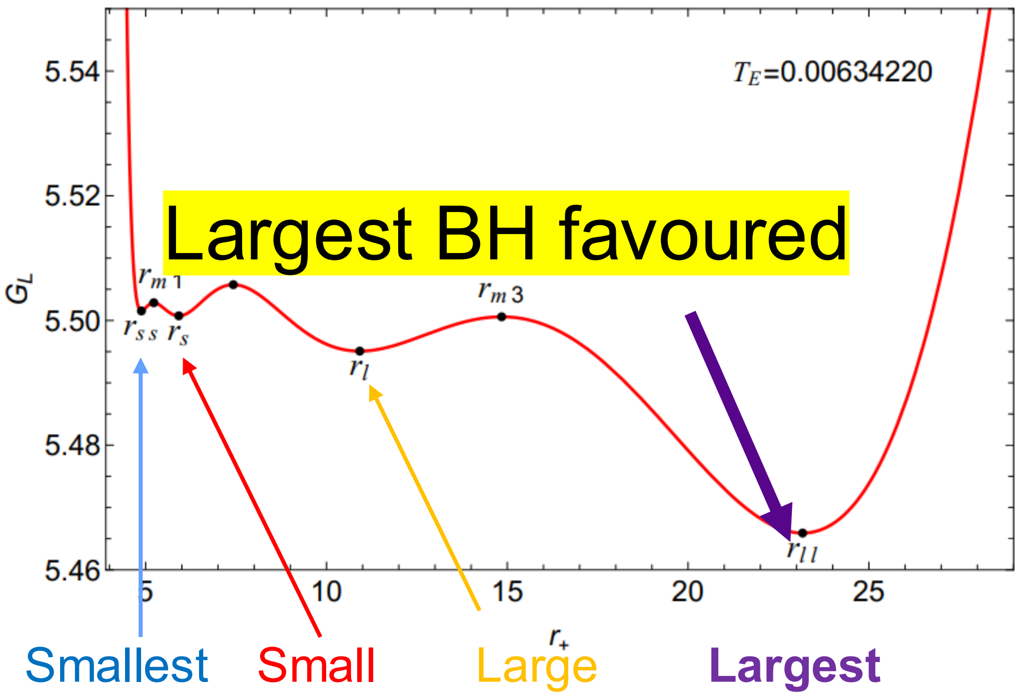

(a) Initial largest (b) Initial large

(c) Initial small (d) Initial smallest

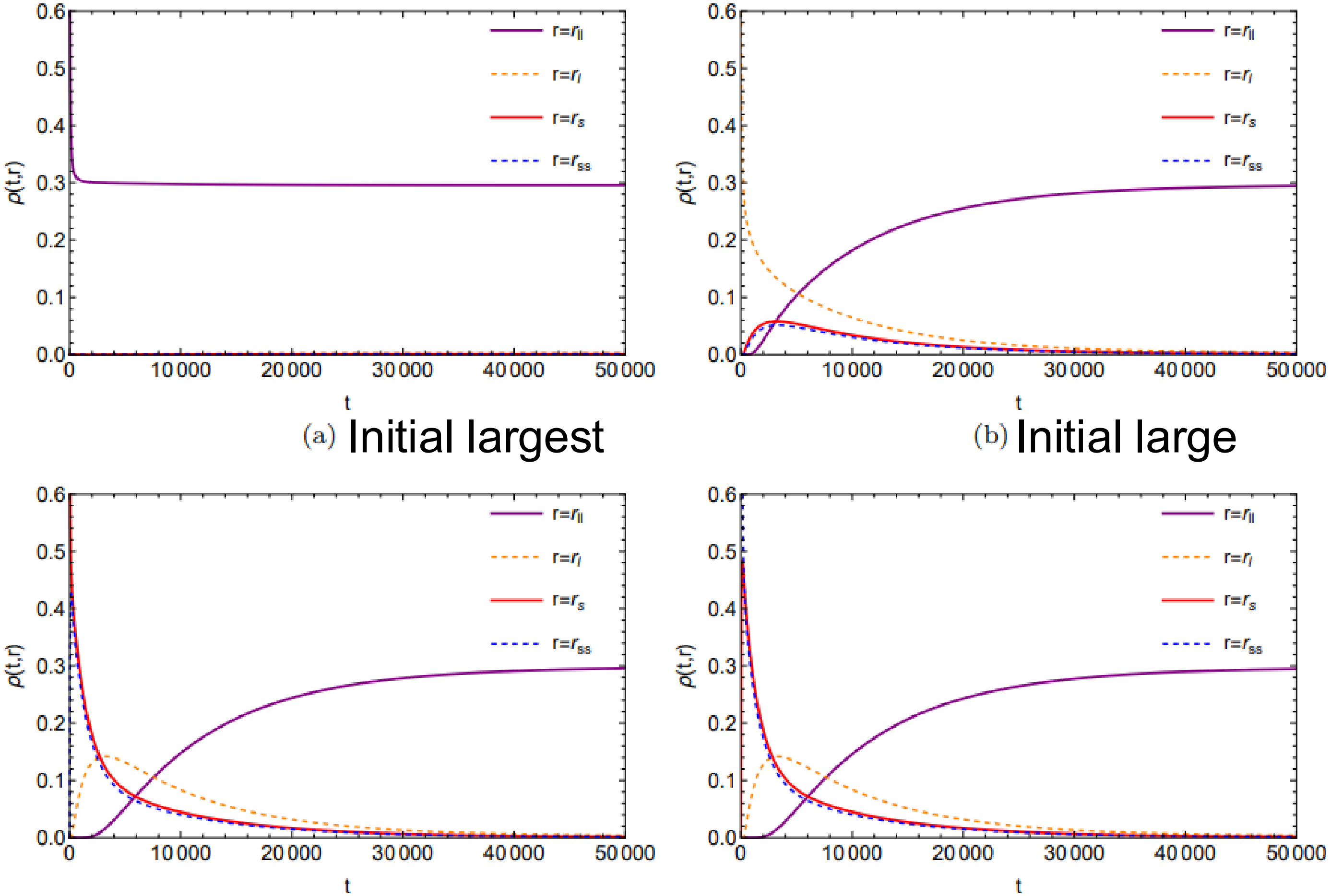
# Probability evolution for 4 phases at high temperature

Conclusion:

At high temperature, the largest BH eventually dominates regardless of the given initial state.




Largest BH (Purple curve) wins!




(a) Initial largest

(b) Initial large

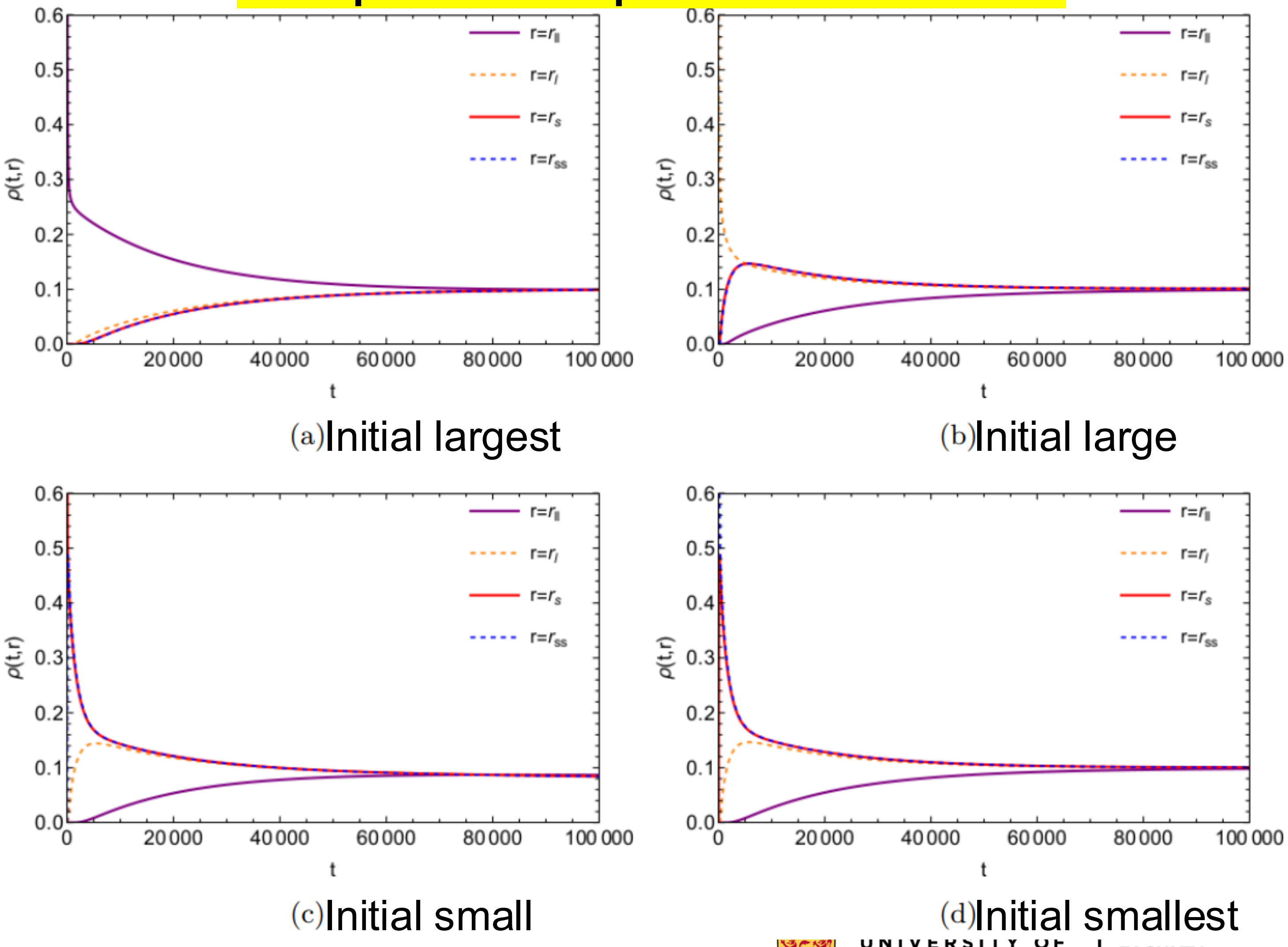
(c) Initial small

(d) Initial smallest

# Probability evolution for 4 phases at quadruple point temperature

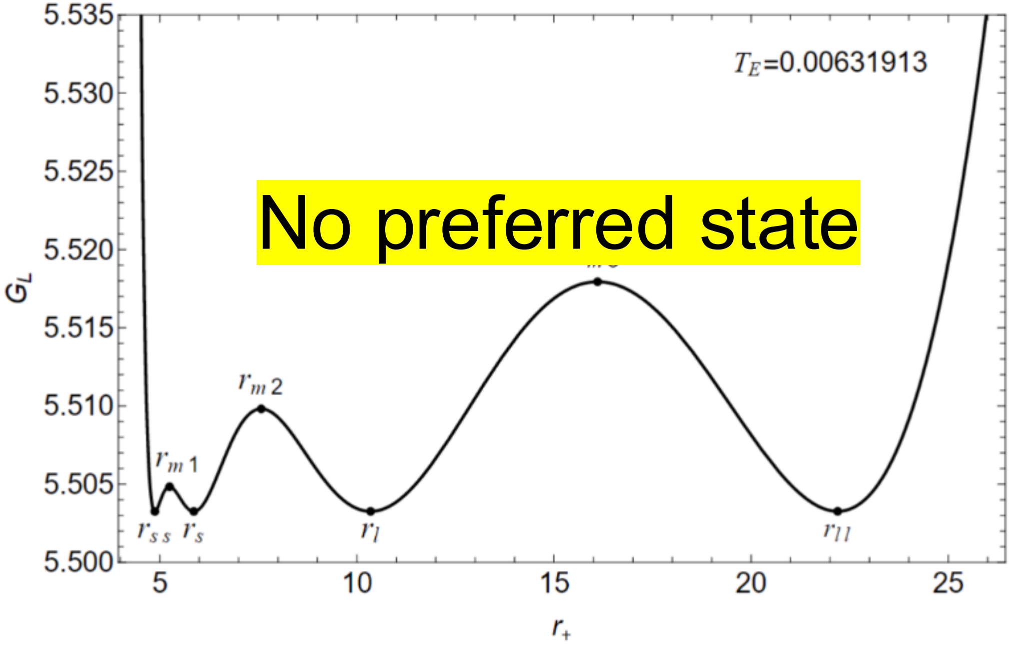
4 equivalent phases coexist!

At quadruple point temperature, four BH phases share identical final stationary probability regardless of the initial state.




(a)Initial largest (b)Initial large

(c)Initial small (d)Initial smallest




(b)

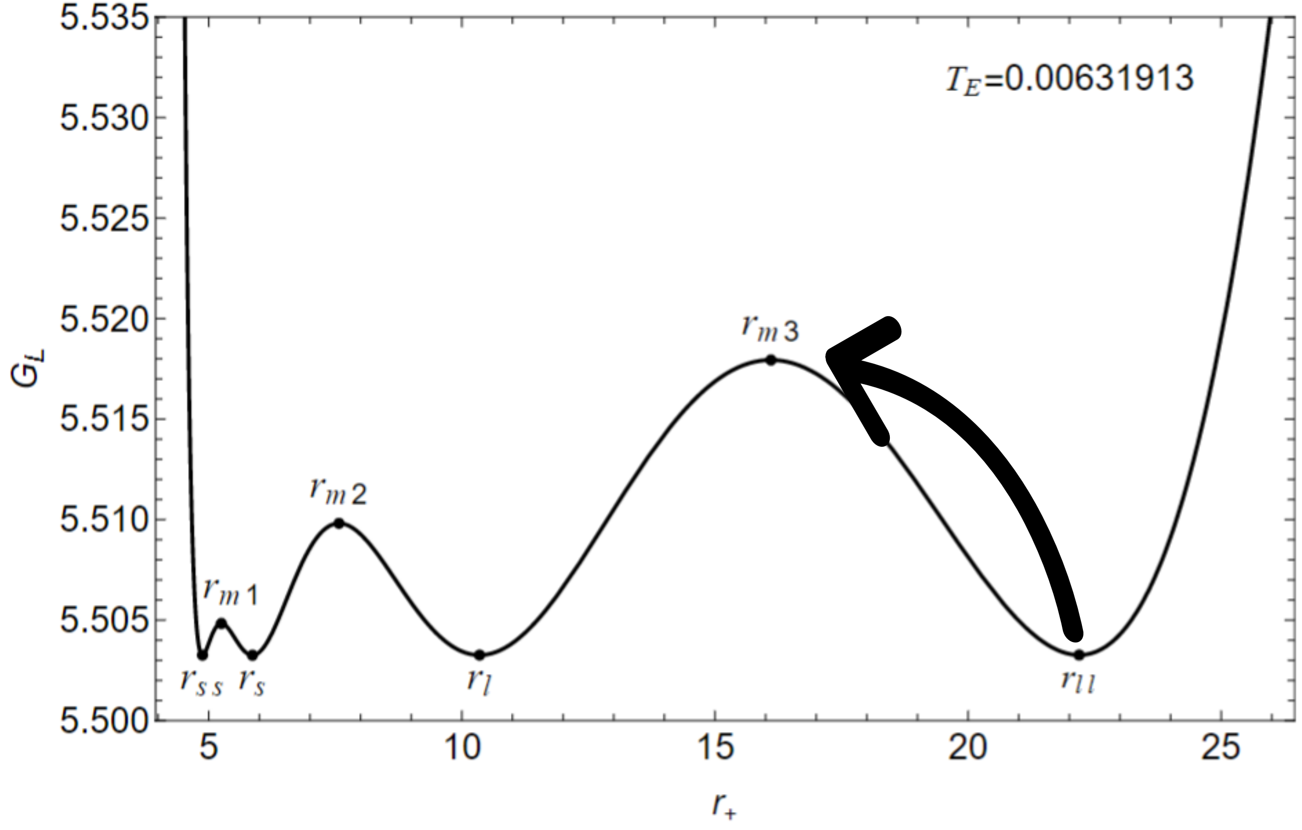
# THE FIRST PASSAGE EVENTS

# First passage events

- First passage time (for a given initial phase):
  The required time to first ascend to the top of nearby free energy barriers that correspond to unstable phases.

- Absorbing boundary conditions

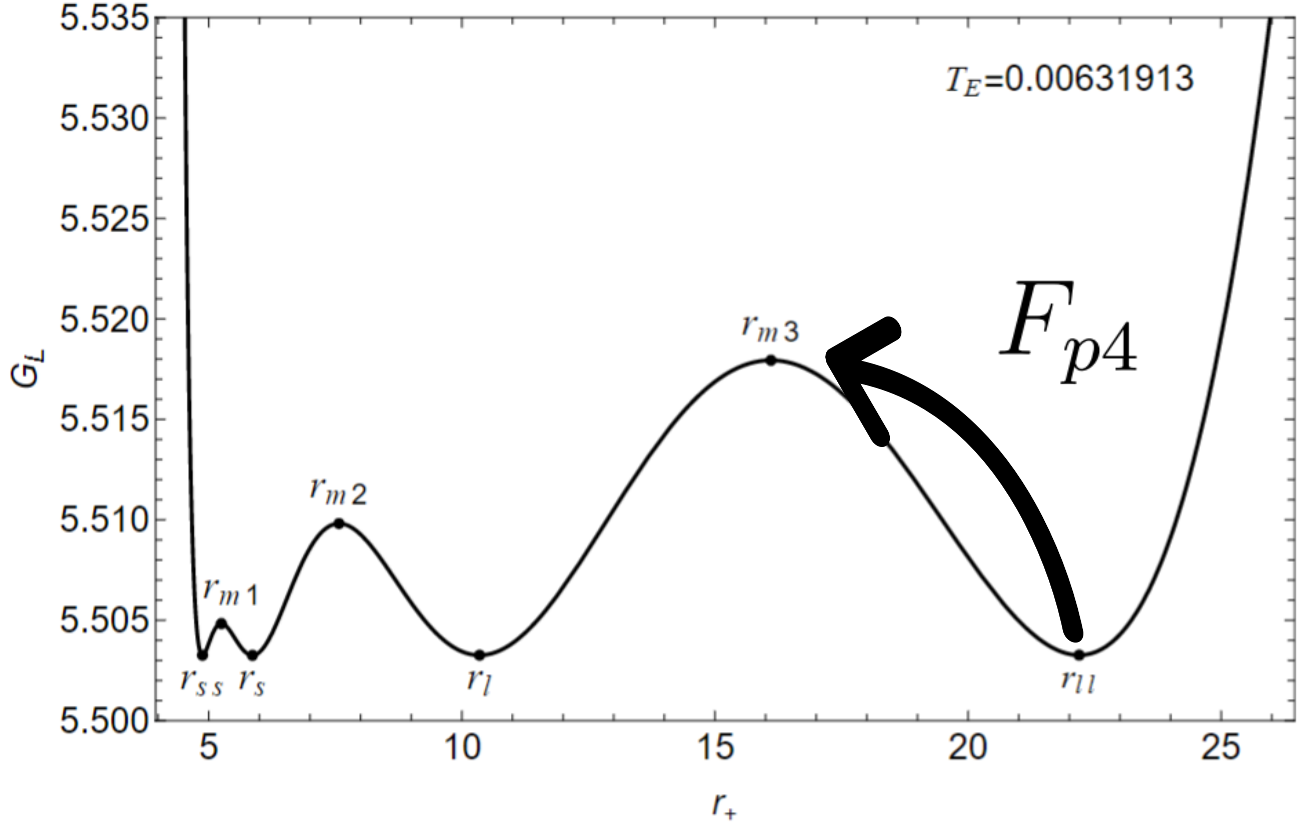
$$\rho(t, r_{m1}) = \rho(t, r_{m2}) = \rho(t, r_{m3}) = 0$$



# First passage events

- First passage time distribution (for a given initial phase):

$$
\begin{aligned}
F_{p1}(t) &= -\frac{\partial \rho(t, r_{m1})}{\partial r}, \\
F_{p2}(t) &= \frac{\partial \rho(t, r_{m1})}{\partial r} - \frac{\partial \rho(t, r_{m2})}{\partial r}, \\
F_{p3}(t) &= \frac{\partial \rho(t, r_{m2})}{\partial r} - \frac{\partial \rho(t, r_{m3})}{\partial r}, \\
F_{p4}(t) &= \frac{\partial \rho(t, r_{m3})}{\partial r}.
\end{aligned}
$$



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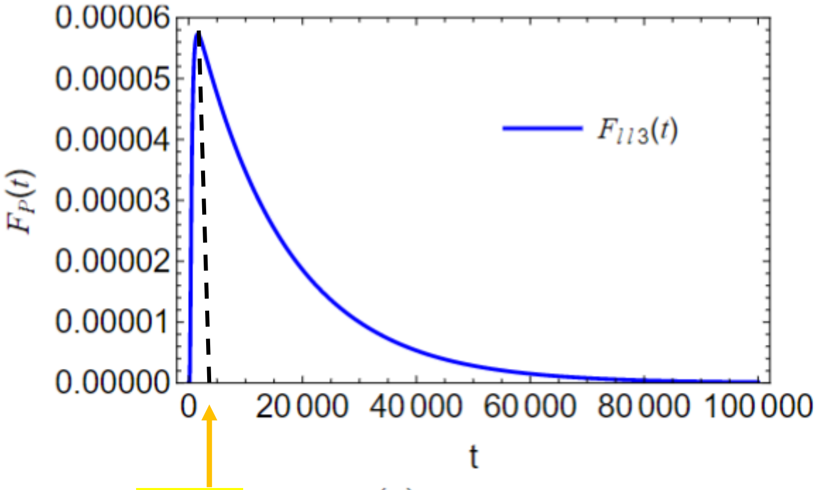
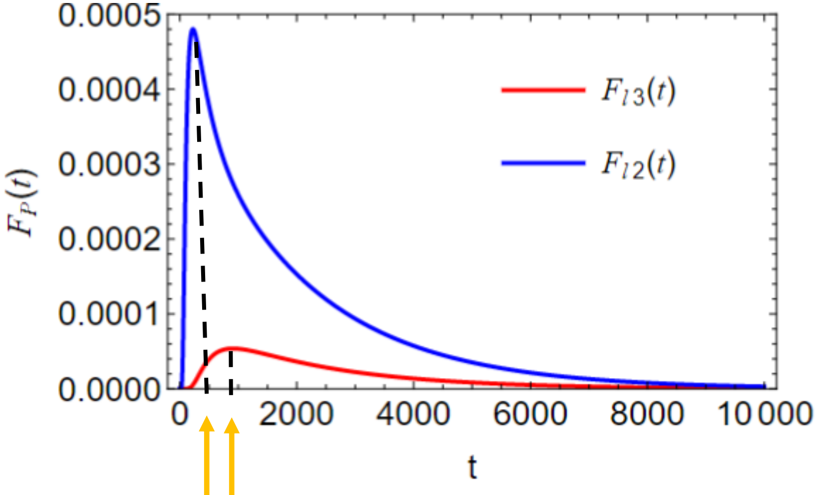
# First passage time distribution




1593

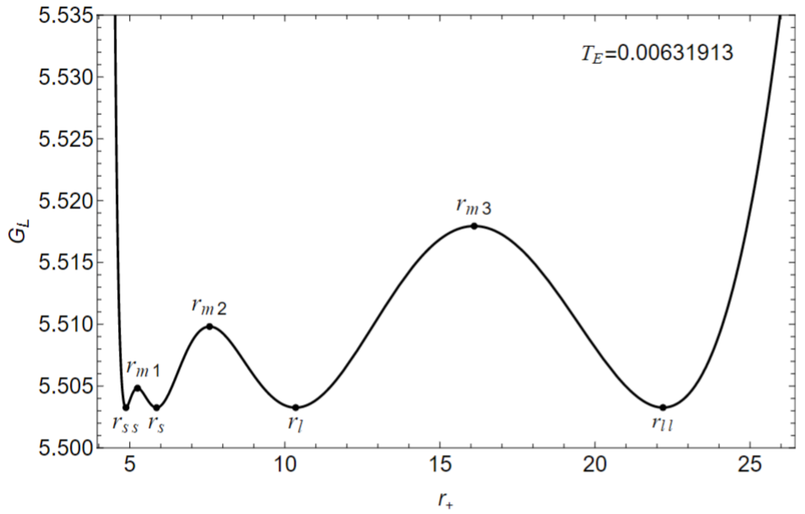
(a) Initial largest




224 912

(b) Initial large

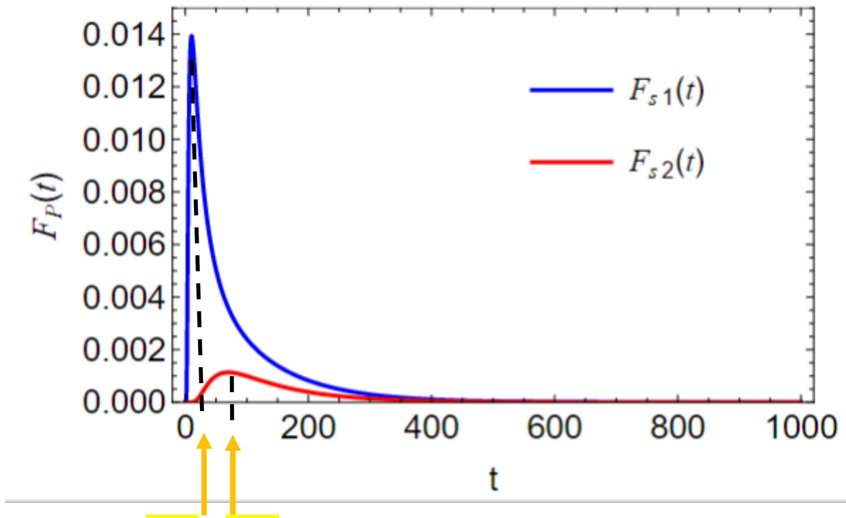
At quadruple point temperature




| | $t_{ll3}$ | $t_{l3}$ | $t_{l2}$ | $t_{s2}$ | $t_{s1}$ | $t_{ss1}$ |
|---|---|---|---|---|---|---|
| $T_L$ | 307 | 1160 | 132 | 81 | 8 | 5 |
| $T_I$ | 1593 | 912 | 224 | 70 | 10 | 4 |
| $T_H$ | 3309 | 466 | 363 | 58 | 13 | 3 |

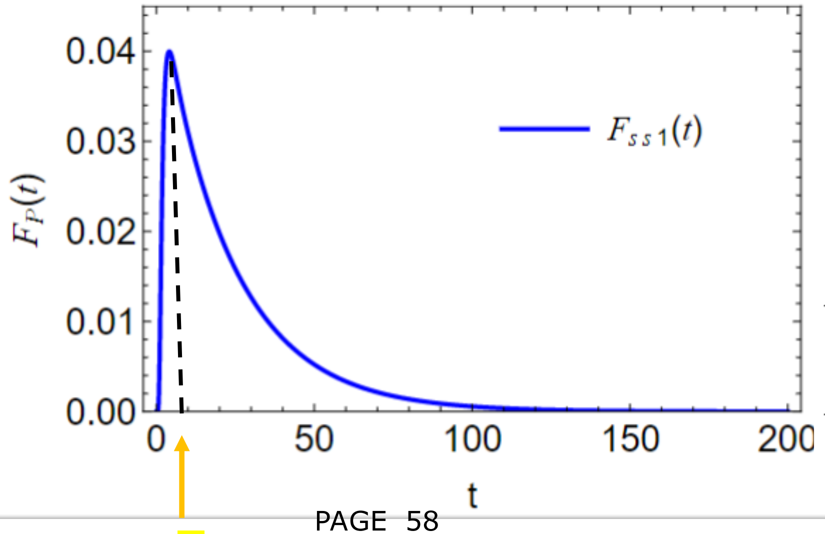
$$t_{ll3} > t_{l3} > t_{l2} > t_{s2} > t_{s1} > t_{ss1}$$

$$W_{ll3} > W_{l3} > W_{l2} > W_{s2} > W_{s1} > W_{ss1}$$




10 70

Initial small




4

Initial smallest

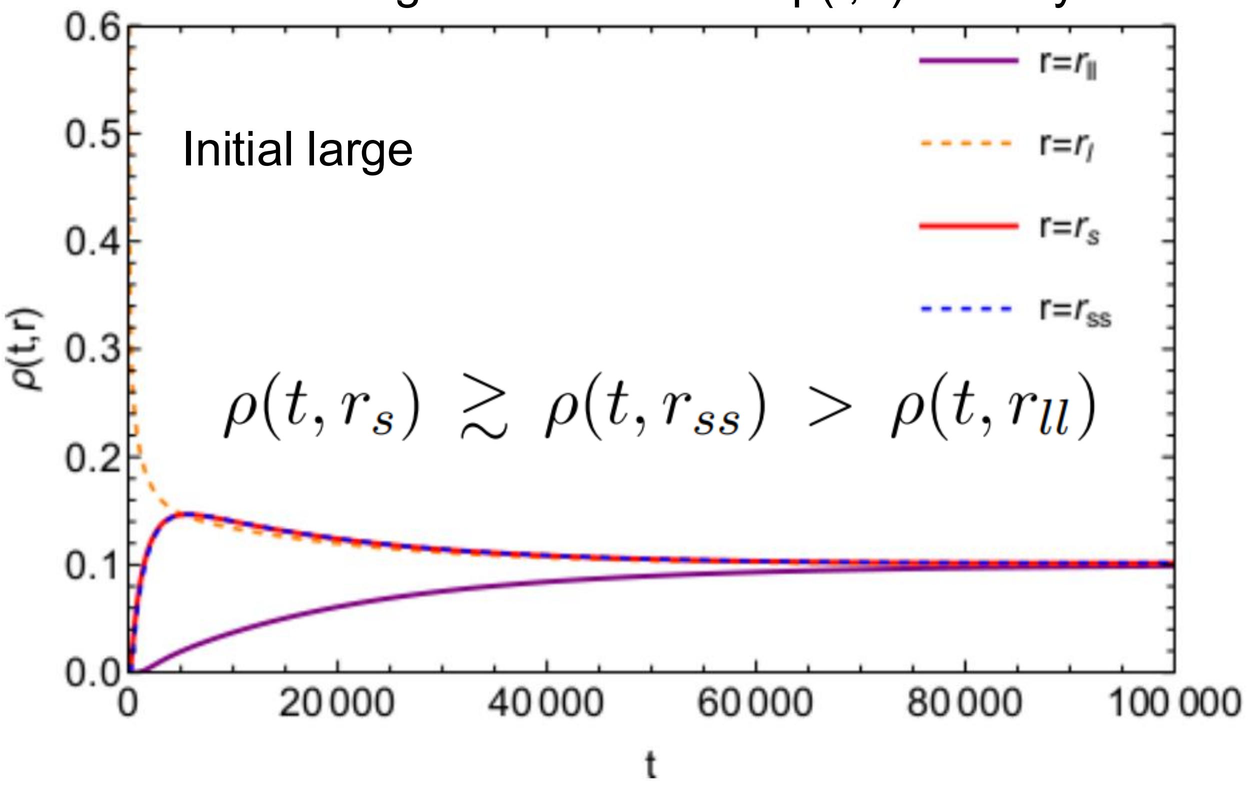
# First passage time distribution

Phase transition time scale

| | $t_{ll3}$ | $t_{l3}$ | $t_{l2}$ | $t_{s2}$ | $t_{s1}$ | $t_{ss1}$ |
|---|---|---|---|---|---|---|
| $T_L$ | 307 | 1160 | 132 | 81 | 8 | 5 |
| $T_I$ | 1593 | 912 | 224 | 70 | 10 | 4 |
| $T_H$ | 3309 | 466 | 363 | 58 | 13 | 3 |

| | $t(l \to s)$ | $t(l \to ss)$ | $t(l \to ll)$ |
|---|---|---|---|
| $T_L$ | 132 | 140 | 1160 |
| $T_I$ | 224 | 234 | 912 |
| $T_H$ | 363 | 376 | 466 |

Understanding the behaviour of ρ(t, r) at early times.




$$\rho(t, r_s) \gtrsim \rho(t, r_{ss}) > \rho(t, r_{ll})$$

$$t(l \to s) \lesssim t(l \to ss) < t(l \to ll)$$

# CONCLUSION

# Summary

- Black hole multicritical points exist!
- Phase transitions can happen at multicritical points.
- Dynamics of BH phase transitions now being understood.
  - Initially, the probabilities will leak to other states from the initial state, indicating the occurrence of the phase transition.
  - Finally, the probability distribution will become stationary, which is determined by the generalized Gibbs free energy $G_L$.

# Acknowledgments


- We would like to thank Jerry Wu, Si-Jiang Yang, Shao-Wen Wei, Yu-Xiao Liu, Yong-Qiang Wang, and Niayesh Afshordi for helpful discussions.
- This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.
- J.Y. is grateful for support from a Mitacs Globalink Graduate Fellowship and encouragements from Hai-jun Wang and Han-qing Liu.
- Thank Yi-pin Jing for providing me with a keyboard to revise the slides!

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# Thank you for your attention!

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# APPENDIX

# 1

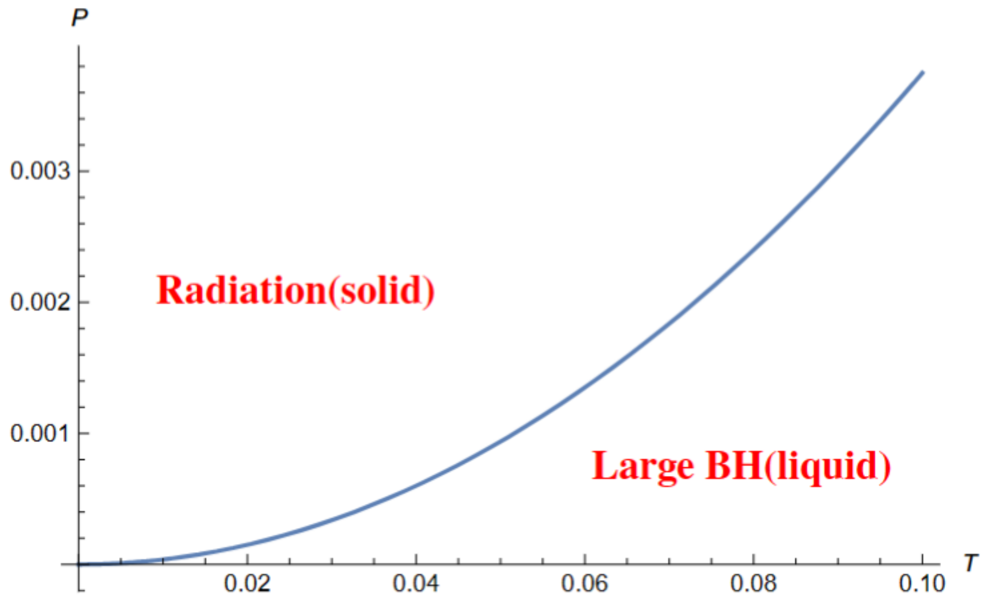
# Introduction

- Black Hole Phase Transition----Hawking–Page transition

Pressure

$$P|_{coex} = \frac{3\pi}{8}T^2$$

Pressure vs temperature phase diagram




Coexistence line of radiation/large BH is similar to solid/liquid coexistence line

Kubizňák, D., Mann, R. B., & Teo, M. (2017). Black hole chemistry: thermodynamics with Lambda. *Classical and Quantum Gravity*, *34*(6), 063001.

# Enthalpy

- $H = E + PV$
- Enthalpy=the energy to create the BH +the energy to place the object with volume V in the environment that has pressure P
- Positive $\Lambda$, cosmic tension, (fluid that has negative pressure) dS!
- Negative $\Lambda$, cosmic pressure (fluid that has positive pressure)! AdS!

# NLE

- NLE was historically introduced in the 1930s to remove the divergence of the electron's self-energy in classical electrodynamics

- The Maxwell action is quadratic in $F_{\mu\nu}$ and Maxwell's equations are linear.

- But Maxwell's electrodynamics may receive non-linear corrections of higher orders in $F_{\mu\nu}$ and may thus be extended to a non-linear theory.

- The Principle of Superposition is the sum of two or more solutions is also a solution for linear differential equation!

# How to adjust parameters to find quadruple point?

- By carefully choosing the locations of the local extrema in $T(r+)$, it is possible to produce a quadruple point.
- Further adjusting the locations of the extrema, the inflections can be made to occur at the same temperature, resulting in the quadruple point.

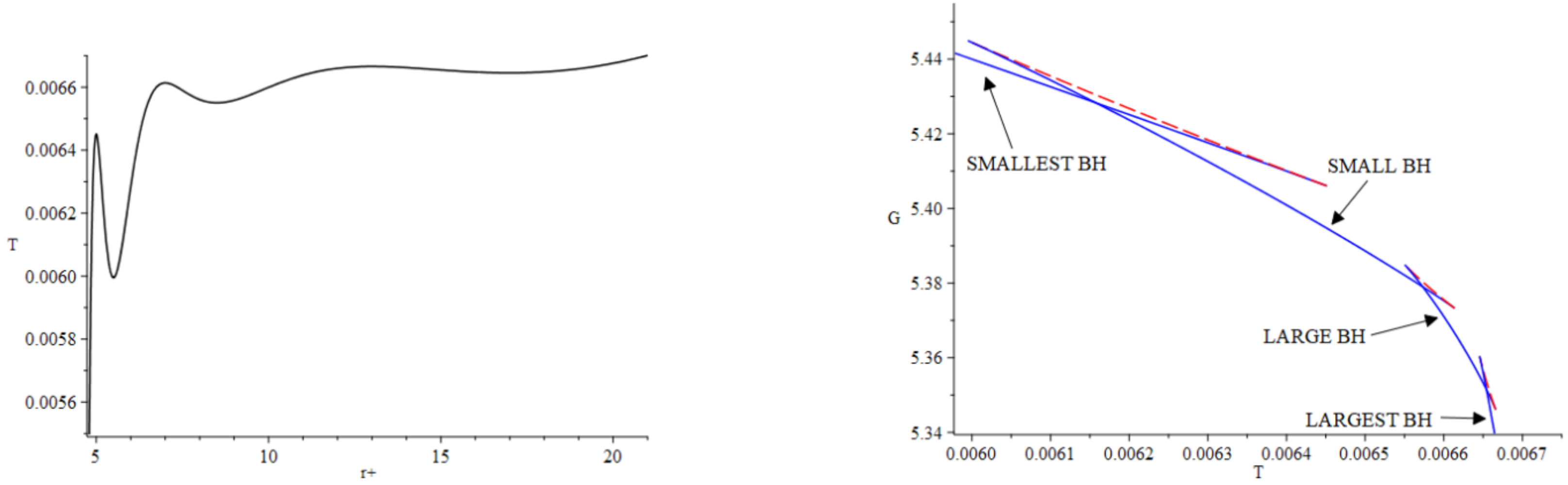
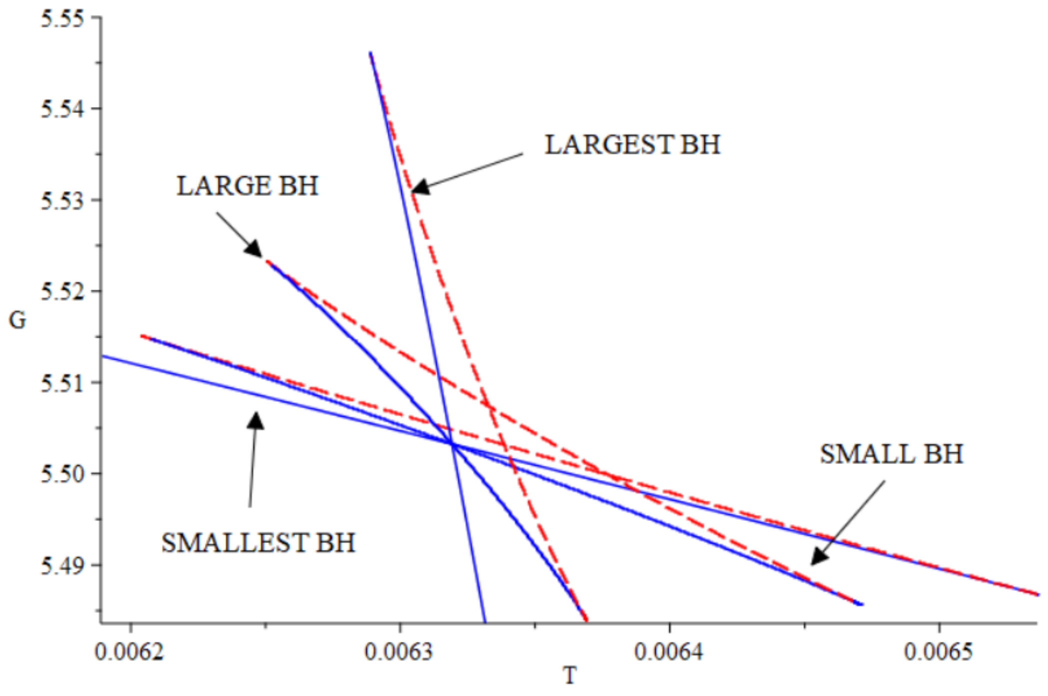
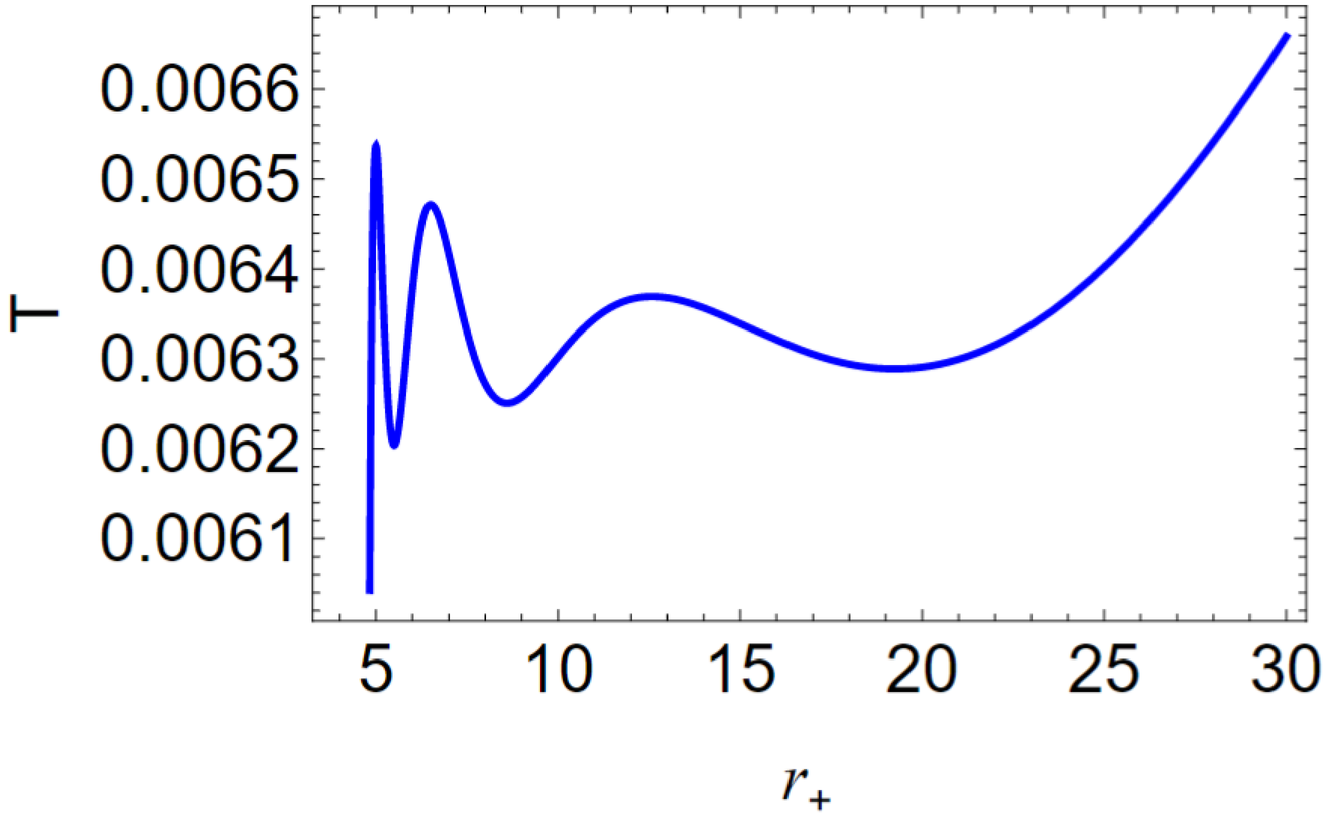



FIG. 1. **Three stable first order phase transitions.** $P = 7.82 \times 10^{-5}, Q = 6.623, \alpha_2 = -21.63694203, \alpha_3 = 1493.535254, \alpha_4 = -148046.3896, \alpha_5 = 1.759261993 \times 10^7, \alpha_6 = -2.332423991 \times 10^9, \alpha_7 = 7.686192644 \times 10^{21}$. *Left.* Inflections in $T(r_+)$ occur at increasing temperatures. *Right.* Three separated swallowtails in the Gibbs free energy, indicating three first order phase transitions.

# Quadruple points in black hole phase transitions




T ∈ (0.00628883, 0.00636918)
N-tuple critical points
(N − 1) swallowtails
(N − 1) pairs of local extrema

# Smoluchowski equation

stays in a specific black hole state with horizon radius $r$ at time $t$. According to the free energy landscape, black hole systems can experience thermodynamic phase transitions and transition to different black hole states via a thermal fluctuation of the order parameter $r$. This process can be described by a particular case of the Fokker-Planck equation, known as the Smoluchowski equation [72],

$$\frac{\partial \rho(t,r)}{\partial t} = D\frac{\partial}{\partial r}\left(e^{-\beta G_{\mathrm{L}}(r)}\frac{\partial}{\partial r}\left(e^{\beta G_{\mathrm{L}}(r)}\rho(t,r)\right)\right), \tag{3.12}$$

where $G_{\mathrm{L}}(r)$ represents the off-shell Gibbs free energy acting as the potential, $\beta$ is the inverse ensemble temperature, $D = k_{\mathrm{B}}T_{\mathrm{E}}/\zeta$ is the diffusion coefficient, with $k_{\mathrm{B}}$ being Boltzmann's constant and $\zeta$ the dissipation coefficient. To investigate the dynamic evolution of various black hole phases near the quadruple point, we numerically solve the Smoluchowski equation to find the probability distribution function $\rho(t,r)$. In the solving procedure we set $k_{\mathrm{B}} = \zeta = 1$ without loss of generality. Solving this partial differential

The kinetic process of black hole phase transition is a stochastic process caused by thermal fluctuation

# Example

- Asymptotically flat, BH has Hawking radiation
- Asymptotically AdS(has positive pressure, like a box)

Hawking radiation will be reflected back to the BH, you can have a static BH in **thermal equilibrium** with its own radiation

- The thermodynamic potential of interest is the *Gibbs free energy*

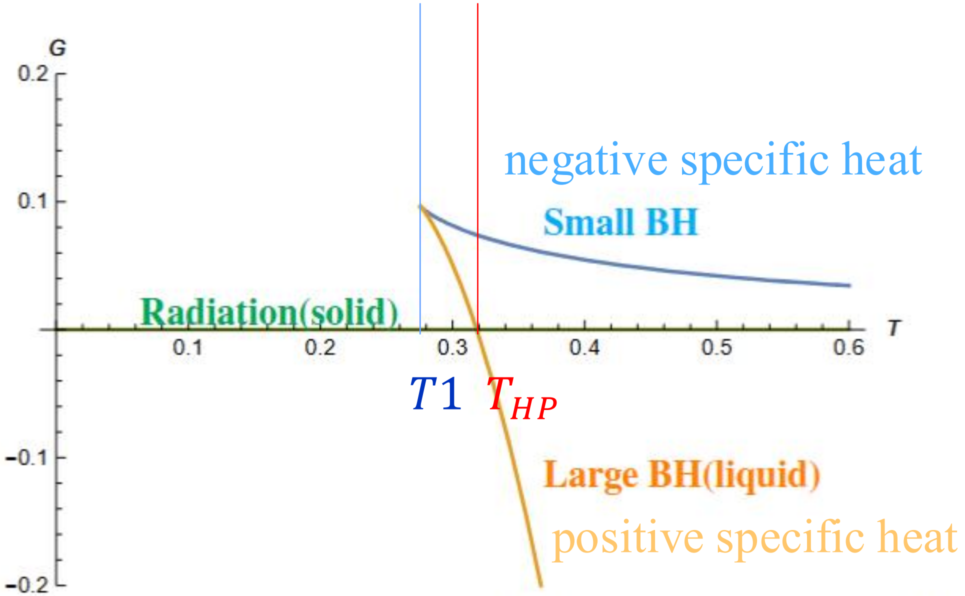
$$G = M - TS = G(P, T, J_1, \ldots, J_N, Q_1, \ldots, Q_n)\,. \tag{4.1}$$

  The equilibrium state corresponds to the global minimum of $G$.

- *Local thermodynamic stability* corresponds to positivity of the specific heat

$$C_P \equiv C_{P,J_1,\ldots,J_N,Q_1,\ldots,Q_n} = T\left(\frac{\partial S}{\partial T}\right)_{P,J_1,\ldots,J_N,Q_1,\ldots,Q_n}\,. \tag{4.2}$$

# Example




first-order phase transition

**Figure 3.1:** Hawking–Page transition (Blue curve is small BH, orange curve is large BH, Green curve is radiation) at fixed $P = \frac{3}{8\pi}$

1. $T \leq T_1$(the intersection of Small BH and Large BH), no black hole can exist. It is just the thermal AdS spacetime.
2. $T_1 \leq T \leq T_{HP}$ Black holes are unstable and will emit Hawking radiation to gradually change to thermal AdS spacetime.
3. $T \geq T_{HP}$ Preferred phase are large black holes.There is a first-order phase transition between thermal radiation and large black holes.

# Example

### 3.1.1 Hawking–Page transition between the thermal radiation and large black holes

Uncharged Black hole solution(dim=3+1)

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \tag{3.3}$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{r^2}{l^2} \tag{3.4}$$

$$f(r) = k - \frac{2M}{r} + \frac{r^2}{l^2} \tag{3.5}$$

with $k = 1$ being the (d − 2)-sphere(spherical horizon geometry), $k = 0$ being a torus(planar horizon geometry), and $k = 1$ being a compact hyperbolic space (hyperbolic horizon geometry)

Set $f(r) = 0$, we find the largest zero of $f(r)$, denoted it by $r_+$ So we have

$$f(r_+) = k - \frac{2M}{r_+} + \frac{r_+^2}{l^2} = 0 \tag{3.6}$$

Solve this, we get

$$M = \frac{r_+}{2}(k + \frac{r_+^2}{l^2}) \tag{3.7}$$

Actually

$$M = \frac{A_k r_+}{8}(k + \frac{r_+^2}{l^2}) \tag{3.8}$$

where $A_{k=1} = 4$ for the spherical case

The entropy $S$ is

$$S = \frac{A}{4} = \frac{\pi r_+^2 A_k}{4} \tag{3.9}$$

The pressure $P$ is given by

$$P = \frac{(d-1)(d-2)}{16\pi l^2} = \frac{3}{8\pi l^2} \tag{3.10}$$

The volume is

$$V \equiv \left(\frac{\partial M}{\partial P}\right)_{S,Q,J} = \frac{\partial}{\partial P}\left(\frac{A_k r_+}{8}(k + \frac{8\pi P r_+^2}{3})\right) = \frac{A_k \pi r_+^3}{3} \tag{3.11}$$

Hence the temperature is

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi}(\frac{2M}{r^2} + \frac{2r}{l^2})|r = r_+ \tag{3.12}$$

$$= \frac{1}{4\pi}(\frac{2Ml^2 + 2r_+^3}{l^2 r_+^2}) \tag{3.13}$$

$$= \frac{1}{4\pi}(\frac{\frac{A_k}{4}(r_+ k l^2 + r_+^3) + 2r_+^3}{l^2 r_+^2}) \tag{3.14}$$

$$= \frac{\frac{A_k}{4} k l^2 + (\frac{A_k}{4} + 2) r_+^2}{4\pi l^2 r_+} \tag{3.15}$$

$$\tag{3.16}$$

To be specific, we consider the $A_k = 4 (k = 1)$ spherical case,

$$M = \frac{r_+}{2}(k + \frac{r_+^2}{l^2}), \qquad S = \pi r_+^2, \qquad T = \frac{kl^2 + 3r_+^2}{4\pi l^2 r_+}, \qquad V = \frac{4\pi r_+^3}{3} \tag{3.17}$$

$$M = \frac{r_+}{2}(1 + \frac{r_+^2}{l^2}), \qquad S = \pi r_+^2, \qquad T = \frac{l^2 + 3r_+^2}{4\pi l^2 r_+}, \qquad P = \frac{3}{8\pi l^2}, \qquad V = \frac{4\pi r_+^3}{3} \tag{3.18}$$

Gibbs free energy

$$G = M - TS \tag{3.19}$$

$$= \frac{r_+}{2} + \frac{r_+^3}{2l^2} - \frac{r_+}{4}(1 + 3\frac{r_+^2}{l^2}) \tag{3.20}$$

$$= \frac{2l^2 r_+^2 + 2r_+^4 - l^2 r_+^2 - 3r_+^4}{4l^2 r_+} \tag{3.21}$$

$$= \frac{l^2 r_+^2 - r_+^4}{4l^2 r_+} \tag{3.22}$$

$$= \frac{l^2 r_+ - r_+^3}{4l^2} \tag{3.23}$$

$$\tag{3.24}$$

For the expression of temperature $T$ (3.18), solve for $r_+$, we get the radius for large BH

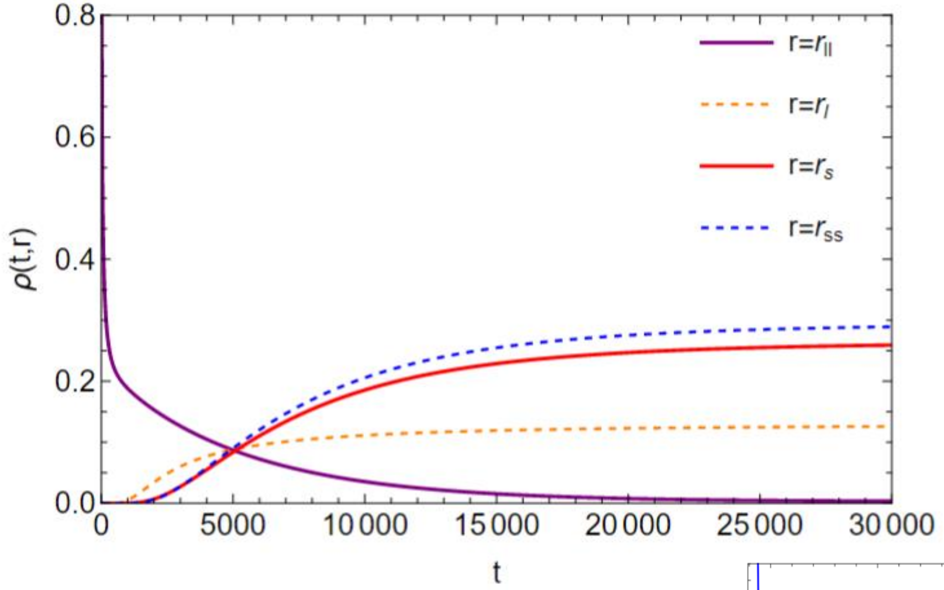
$$r_+ = \frac{1}{6}(4\pi T + \sqrt{-12 + 16\pi^2 T^2}) \tag{3.25}$$

the radius for small BH

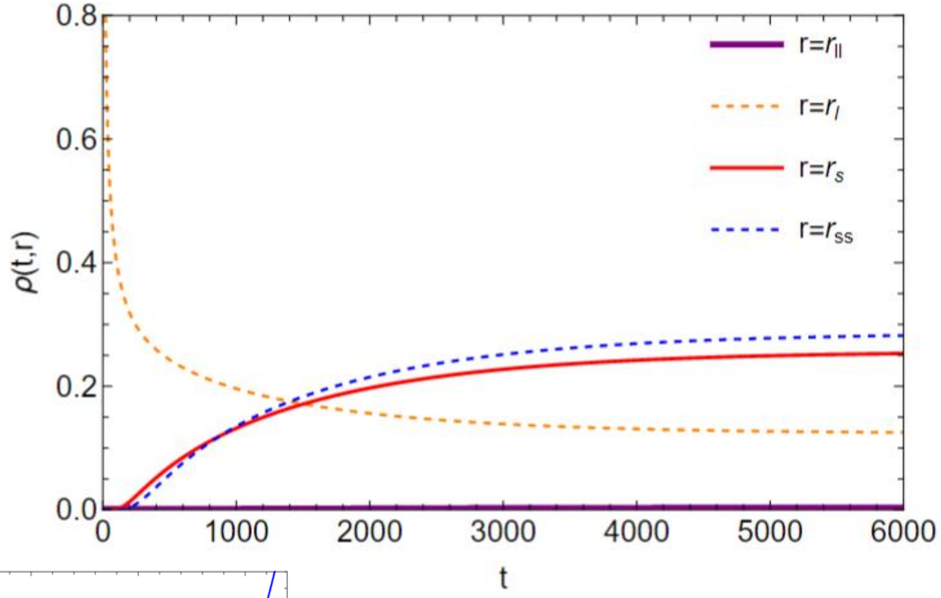
$$r_+ = \frac{1}{6}(4\pi T - \sqrt{-12 + 16\pi^2 T^2}) \tag{3.26}$$

# Probability evolution for four stable phases at low temperature

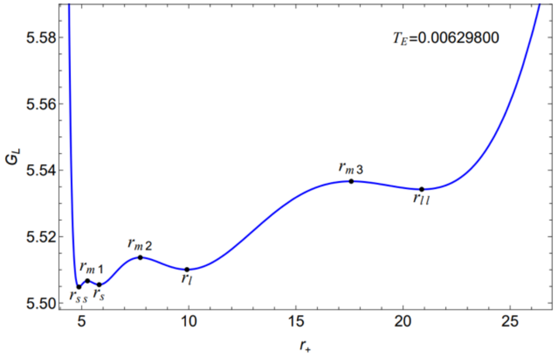
- Initial state is the largest BH




Initial state is the large BH



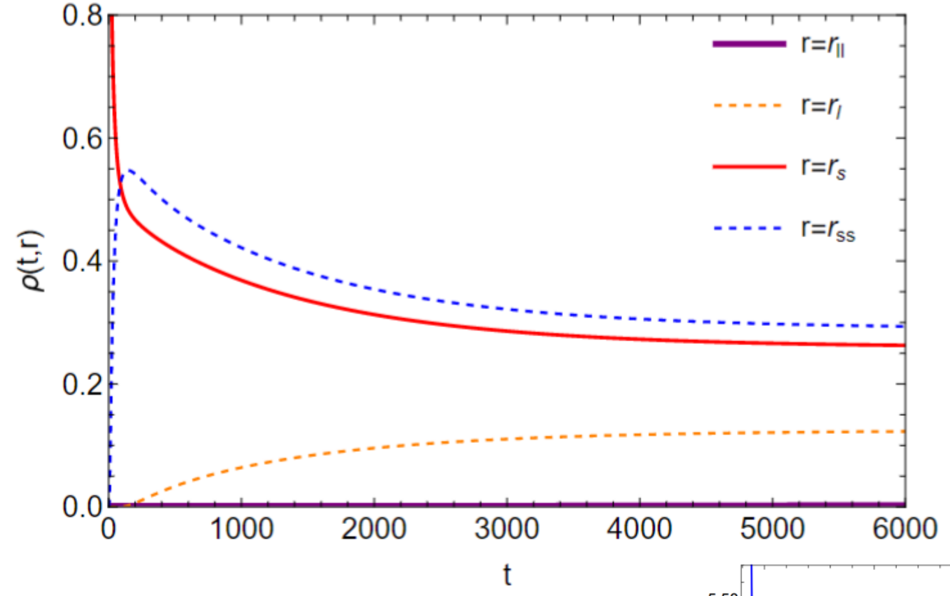

| | $t(ll \to l)$ | $t(ll \to s)$ | $t(ll \to ss)$ |
|---|---|---|---|
| $T_L$ | 307 | 439 | 447 |
| $T_I$ | 1593 | 1817 | 1827 |
| $T_H$ | 3309 | 3672 | 3685 |




| | $t(l \to s)$ | $t(l \to ss)$ | $t(l \to ll)$ |
|---|---|---|---|
| $T_L$ | 132 | 140 | 1160 |
| $T_I$ | 224 | 234 | 912 |
| $T_H$ | 363 | 376 | 466 |

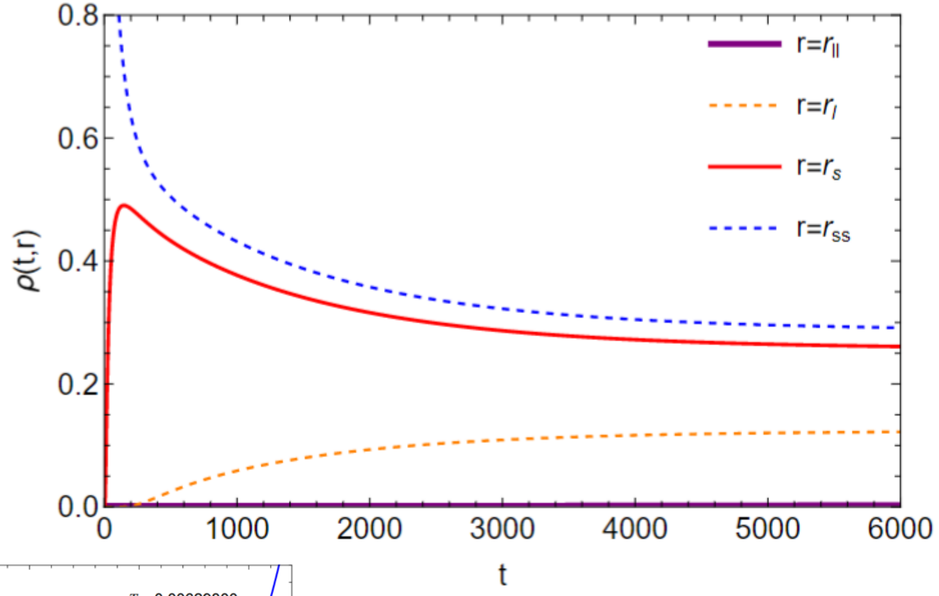
# Probability evolution for four stable phases at low temperature

- Initial state is the small BH

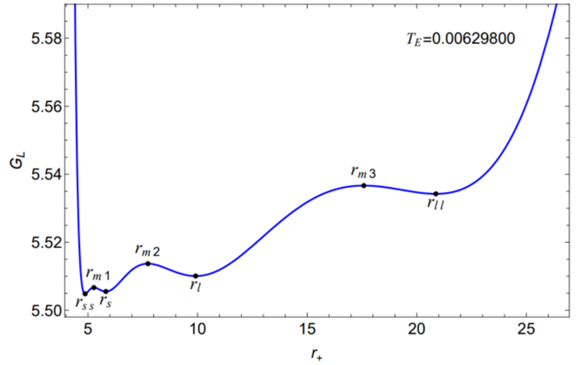



| | $t(s \to ss)$ | $t(s \to l)$ | $t(s \to ll)$ |
|---|---|---|---|
| $T_L$ | 8 | 81 | 1241 |
| $T_I$ | 10 | 70 | 982 |
| $T_H$ | 13 | 58 | 524 |

Initial state is the smllest BH



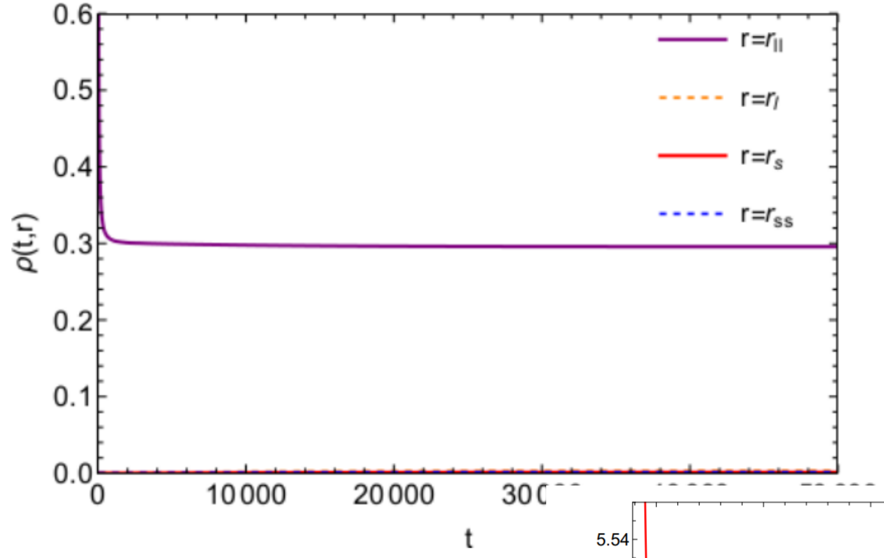

| | $t(ss \to s)$ | $t(ss \to l)$ | $t(ss \to ll)$ |
|---|---|---|---|
| $T_L$ | 5 | 86 | 1246 |
| $T_I$ | 4 | 74 | 986 |
| $T_H$ | 3 | 61 | 527 |

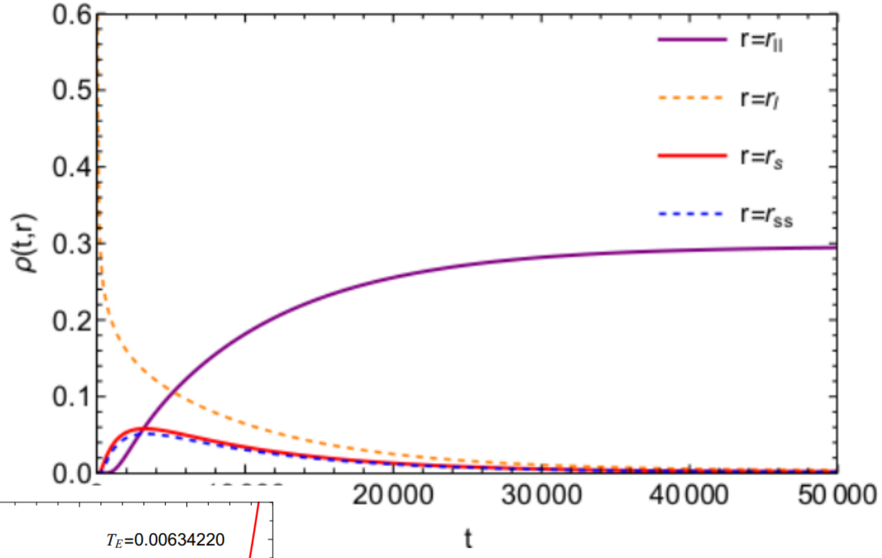


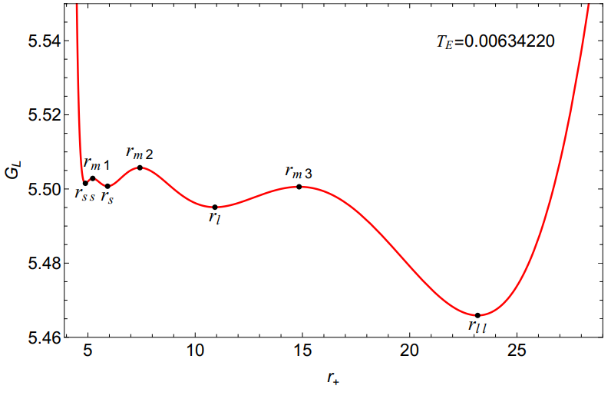
# Probability evolution for four stable phases at high temperature

- Initial state is the largest BH

Initial state is the large BH



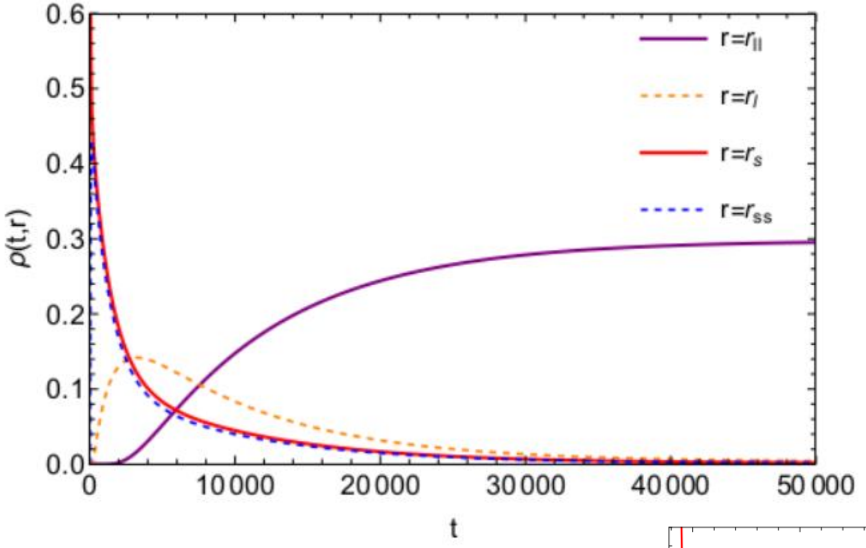

| | $t(s \to ss)$ | $t(s \to l)$ | $t(s \to ll)$ |
|---|---|---|---|
| $T_L$ | 8 | 81 | 1241 |
| $T_I$ | 10 | 70 | 982 |
| $T_H$ | 13 | 58 | 524 |

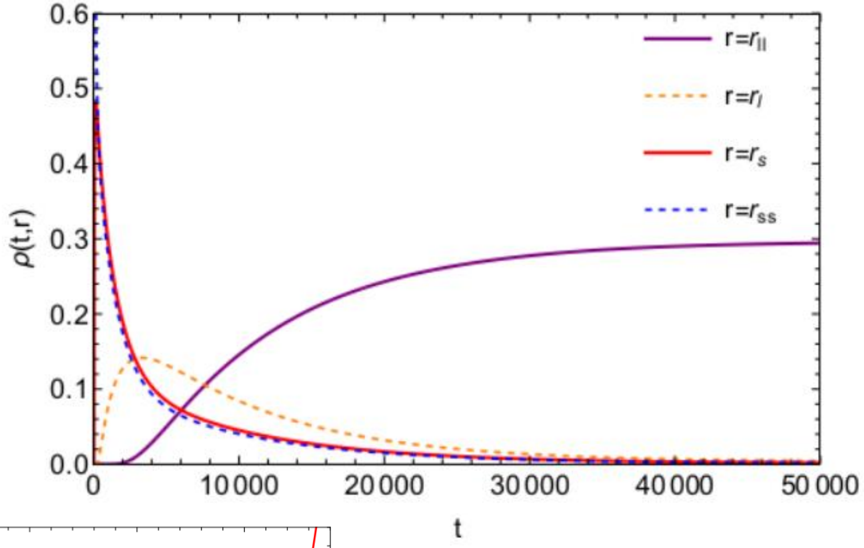
| | $t(ss \to s)$ | $t(ss \to l)$ | $t(ss \to ll)$ |
|---|---|---|---|
| $T_L$ | 5 | 86 | 1246 |
| $T_I$ | 4 | 74 | 986 |
| $T_H$ | 3 | 61 | 527 |

# Probability evolution for four stable phases at high temperature

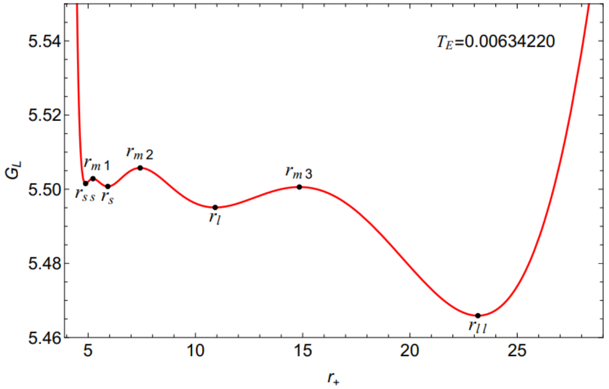
- Initial state is the small BH

Initial state is the smllest BH




| | $t(s \to ss)$ | $t(s \to l)$ | $t(s \to ll)$ |
|---|---|---|---|
| $T_L$ | 8 | 81 | 1241 |
| $T_I$ | 10 | 70 | 982 |
| $T_H$ | 13 | 58 | 524 |




| | $t(ss \to s)$ | $t(ss \to l)$ | $t(ss \to ll)$ |
|---|---|---|---|
| $T_L$ | 5 | 86 | 1246 |
| $T_I$ | 4 | 74 | 986 |
| $T_H$ | 3 | 61 | 527 |

# 2

# MORE ABOUT PHASE TRANSITION

More about phase transition

# Order parameter

ferromagnetic /paramagnetic phase transition

Ordered magnetic moments (ferromagnetic, Figure 1) below the Curie temperature.
$M(T) \neq 0$

The order parameter is the magnetization $M(T)$ that goes from a finite value to zero when the temperature is increased above the Curie temperature.

Disordered magnetic moments (paramagnetic, Figure 2) above the Curie temperature.
$M(T) = 0$

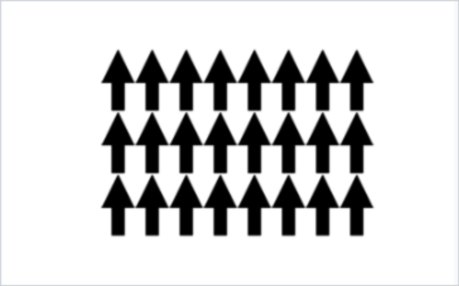


**Figure 1.** Below the Curie temperature, neighbouring magnetic spins align parallel to each other in a ferromagnet in the absence of an applied magnetic field.

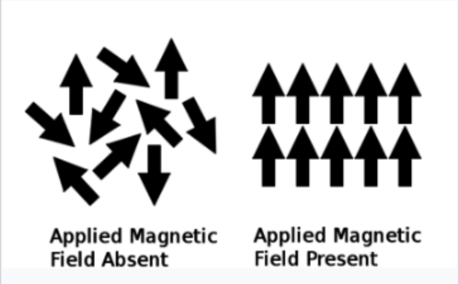



**Figure 2.** Above the Curie temperature, the magnetic spins are randomly aligned in a paramagnet unless a magnetic field is applied.

https://en.wikipedia.org/wiki/Curie_temperature

# CONFINEMENT

Confinement

# Quark confinement

1. Originally, due to **quark confinement**, an infinite amount of energy is required to isolate a single quark, so there are no free quarks.
2. However, at sufficiently high temperatures, a **quark-gluon plasma** can form, in which quarks can move freely to some extent. This is a **deconfining phase**.
3. The process of heating corresponds to the confinement/deconfinement phase transition.

# Gravity dual of confinement

1. In the gravitational duality, physicists map the potential between the quark-antiquark to a string in the bulk.
2. The two ends of the string are on the boundary, corresponding to the quark and the antiquark.
3. If there's **a horizon(Poincare AdS),** the string can drop down arbitrarily close to the horizon as the ends move apart from each other. Because of the infinite gravitational redshift, that costs essentially zero energy, **deconfining phase**.
4. If there is **no horizon(soliton),** the string just stretches along the normal geometry at the minimum radius. The string's energy is proportional to the distance between the ends. To separate the ends infinitely far apart requires infinite energy, corresponding to the **confining phase**.

# Soliton no horizon, Poincare AdS horizon

1. The AdS soliton doesn't have a horizon at $r_0$. ($r_0$ is just an origin in the $r-\phi$ plane). You can tell it's not a horizon because the metric component $g_{tt} \neq 0$ there. So all the AdS solitons (charged and uncharged) are confining.

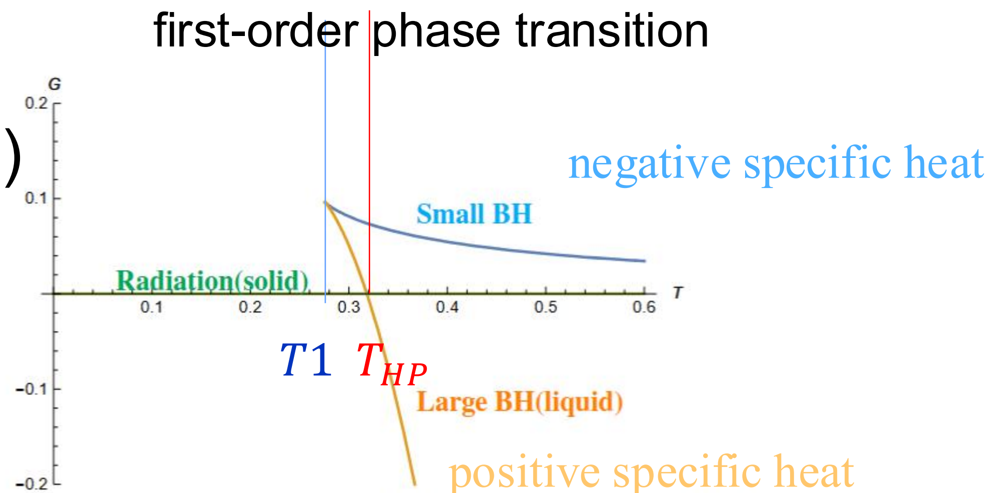
$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + f(r)d\phi^2\right) + \frac{l^2}{r^2 f(r)}dr^2, \qquad f(r) \equiv 1 - \frac{\mu l^2}{r^d} - \frac{Q^2 l^2}{r^{2d-2}}$$

2. We are working in Poincare coordinates, where AdS looks like a radius and Minkowski spacetime (in our case with a periodic direction), so the boundary is Minkowski. Even empty/vacuum AdS has a horizon at r=0 in Poincare coordinates.

$$ds^2 = \frac{r^2}{l^2}\left(-dt^2 + d\vec{x}^2 + d\phi^2\right) + \frac{l^2}{r^2}dr^2, \qquad A = -\frac{\Phi}{\Delta\phi}d\phi$$

# Hawking-Page transition

1. Hawking-Page transition (thermal AdS/AdS black hole) is also a confinement/deconfinement transition




2. AdS in global coordinates, no horizon (confining)

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2\,d\Omega_{n-2}^2$$

where $f(r) = 1 + \dfrac{r^2}{\alpha^2}$

1. Black hole has a horizon (deconfining)

**Figure 3.1:** Hawking–Page transition (Blue curve is small BH, orange curve is large BH, Green curve is radiation) at fixed $P = \frac{3}{8\pi}$

1. $T \leq T_1$(the intersection of Small BH and Large BH), no black spacetime.
2. $T_1 \leq T \leq T_{HP}$ Black holes are unstable and will emit Hawking radiation to gradually change to thermal AdS spacetime.
3. $T \geq T_{HP}$ Preferred phase are large black holes.There is a first-order phase transition between thermal radiation and large black holes.

# The significance of my work for QG?

- BH are where QT and GT meet, neither the Q effect nor the G effect can be ignored. Only when we consider Q effect, we have BH thermodynamics.
- Holography helps us to understand the relationship between the quantum theory and gravitational theory.
- Originally complexity is defined in a quantum system, and in holography, it is dual to some geometric/gravitational observables. Complexity helps us to understand the interior of BH.

**Table 1: Modern values for Planck's original choice of quantities**

| Name | Dimension | Expression | Value (SI units) |
|---|---|---|---|
| Planck length | length (L) | $l_P = \sqrt{\frac{\hbar G}{c^3}}$ | $1.616\,255(18) \times 10^{-35}$ m [7] |
| Planck mass | mass (M) | $m_P = \sqrt{\frac{\hbar c}{G}}$ | $2.176\,434(24) \times 10^{-8}$ kg [8] |
| Planck time | time (T) | $t_P = \sqrt{\frac{\hbar G}{c^5}}$ | $5.391\,247(60) \times 10^{-44}$ s [9] |
| Planck temperature | temperature (Θ) | $T_P = \sqrt{\frac{\hbar c^5}{G k_B^2}}$ | $1.416\,784(16) \times 10^{32}$ K [10] |

https://en.wikipedia.org/wiki/Planck_units